1992 Mu Alpha Theta National Convention

Probability and Statistics Topic Test Answers

1. C
2. C
3. C
4. A
5. C
6. B
7. C
8. B
9. D
10. D
11. A
12. D
13. D
14. A
15. A
16. C
17. A
18. A
19. E
20. B
21. C
22. A
23. B
24. B
25. E
26. B
27. B
28. D
29. A
30. B

1. PUT DATA IN ORDER: 3, 3, 3, 4, 4, 5, 6, 7, 7, 9 ∴ MODE IS 3 ∴
MEDIAN IS $\frac{4+5}{2} = 4.5$ ∴ $3 + 4.5 = \boxed{7.5}$

2. TWO INTEGERS WILL HAVE AN ODD PRODUCT IF AND ONLY IF BOTH NUMBERS ARE ODD. HENCE, OUR PROBABILITY IS
$1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{3/4}$

3. $P = (1/2)(1/5) = 1/10 = \boxed{10\%}$

4. $$\frac{\binom{10}{8}\binom{6}{2}}{\binom{7}{4}} = \frac{\frac{10 \cdot 9}{2 \cdot 1} \cdot \frac{6 \cdot 5}{2 \cdot 1}}{\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}} = \frac{10 \cdot 9 \cdot 3}{2 \cdot 7} = \frac{5 \cdot 9 \cdot 3}{7} = \boxed{\frac{135}{7}}$$

5. P = P(rain) P(go outside in rain) + P(no rain) P(go outside in no rain) =
$= (.2)(.1) + (.8)(.8) = .02 + .64 = \boxed{0.66}$

PROBABILITY & STATISTICS

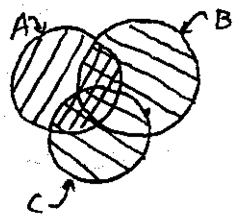
(6) AVERAGE $= \dfrac{1+2+3+\cdots+n}{n} = \dfrac{\frac{n(n+1)}{2}}{n} = \boxed{\dfrac{n+1}{2}}$

(7) WE MUST HAVE $P(A)+P(B)=1$ & $P(A)\geq 0$, $P(B)\geq 0$.
HENCE, $\boxed{(\frac{1}{3}, \frac{2}{3})}$ IS THE ANSWER.

(8) $\dfrac{a+b}{2}=4 \therefore a+b=8 \therefore \dfrac{c+d}{2}=6 \therefore c+d=12 \therefore \dfrac{a+b+c+d}{4} = \dfrac{20}{4} = \boxed{5}$

(9) $B \cup C$ IS

A IS



THE REGION WE SEEK IS , $\boxed{A \cap (B \cup C)}$

(10) CHOOSE 4 COLORS FROM 8 : $\binom{8}{4} = \boxed{70}$.

PROBABILITY & STATISTICS

(11) THERE ARE $8!$ WAYS TO STUFF THE ENVELOPES. THERE ARE $\binom{8}{6}=\binom{8}{2}=28$ WAYS TO CHOOSE 6 ENVELOPES TO STUFF CORRECTLY AND 1 WAY TO CORRECTLY STUFF EACH SET OF SIX. OF THE REMAINING 2 ENVELOPES, THERE IS ONLY ONE WAY TO STUFF THEM INCORRECTLY.

THUS, $P = \dfrac{28}{8!} = \dfrac{1}{2\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = \boxed{\dfrac{1}{1440}}$

(12) $a = \dfrac{9!}{2!} \therefore b = 6! \therefore \dfrac{a}{b} = \dfrac{9!}{2!6!} = \dfrac{9\cdot8\cdot7}{2} = \boxed{252}$

(13) THERE ARE 2 POSSIBILITIES : MFM : $\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)\left(\frac{2}{5}\right) = \frac{12}{125}$

FMF : $\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = \frac{18}{125}$

THUS, TOTAL PROBABILITY IS $\dfrac{12+18}{125} = \boxed{6/25}$

(14) CONSIDER THE SET WHICH HAS 1 AS AN ELEMENT. THE NUMBER OF WAYS TO CHOOSE 4 MORE NUMBERS IS $\binom{9}{4}$. THE NUMBER OF WAYS TO CHOOSE 2 AND THREE OTHER NUMBERS IS $(1)\binom{8}{3}$.

THUS, OUR PROBABILITY IS $\dfrac{\binom{8}{3}}{\binom{9}{4}} = \boxed{\dfrac{4}{9}}$

(15) $P = \dfrac{P(\text{BAG A \& BLACK MARBLE})}{P(\text{BAG A \& BLACK}) + P(\text{BAG B \& BLACK})} = \dfrac{\frac{1}{2}(\frac{2}{3})}{\frac{1}{2}(\frac{2}{3}) + (\frac{3}{8})(\frac{1}{2})} = \dfrac{\frac{1}{3}}{\frac{1}{3}+\frac{3}{16}} =$

$= \dfrac{16/48}{\frac{16+9}{48}} = \boxed{16/25}$

Probability & Statistics

6) $P((A\cap B)\cup C) = P(A\cap B) + P(C) - P(A\cap B\cap C) = P(A)+P(B)-P(A\cup B)+P(C)-P(A\cap B\cap C) =$

$$= 1/5 + 2/5 - 1/2 + 1/4 - 1/20 = 3/5 - 1/4 - 1/20 = 3/5 - 3/10 = \boxed{3/10}$$

17) $\dfrac{(9n+a)^2}{9} = \dfrac{81n^2+18an+a^2}{9} = 9n^2 + 2an + \dfrac{a^2}{9}$. Thus, the remainder depends upon a:

a	0	1	2	3	4	5	6	7	8
remainder	0	1	4	0	7	7	0	4	1

$\boxed{\dfrac{2}{9}}$

18) We consider the integers mod 4: From 1 to 13, there are 3 integers congruent to 0 mod 4, 4 congruent to 1 mod 4, 3 congruent to 2, and 3 congruent to 3. The possible sets of congruencies whose sum is congruent to zero mod 4 are:

$\{0,0,0\} \to \binom{3}{3} = 1$ poss.

$\{2,2,0\} \to \binom{3}{2}\binom{3}{1} = 9$ poss.

$\{1,3,0\} \to \binom{3}{1}\binom{3}{1}\binom{4}{1} = 36$ poss.

$\{3,3,2\} \to \binom{3}{2}\binom{3}{1} = 9$ poss.

$\{1,1,2\} \to \binom{4}{2}\binom{3}{1} = 18$ poss.

Thus, we have a total of 73 ways to choose 3 integers whose sum is divisible by 4. There are $\binom{13}{3} = 286$ ways to choose 3 integers from 13. Thus, $P = \boxed{\dfrac{73}{286}}$

19) The expected income of the game is:

$$(\$0)\left(\tfrac{1}{2}\right) + (\$1)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + (\$2)\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right) + (\$4)\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right) + \cdots = \sum_{n=0}^{\infty}\left(2^n\left(\tfrac{1}{2}\right)^{n+2}\right) =$$

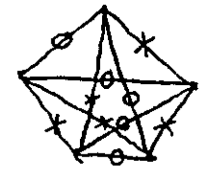
$$= \sum_{n=0}^{\infty}\left(\tfrac{1}{4}\right),$$ which diverges. Theoretically, the man can pay an infinite amount and expect to break even.

20) If the series ends in 6 games, the score must be 3 games to 2 games after 5 games. This can happen in $\binom{5}{3} = 10$ ways. The winning team wins 4 games & the losing team 2. Thus, the probability that a certain team wins in 6 is $10\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^2 = \tfrac{10}{64} = \tfrac{5}{32}$. Since each team is equally likely to win; thus $P = 2\left(\tfrac{5}{32}\right) = \boxed{\dfrac{5}{16}}$

PROBABILITY & STATISTICS

21) FIRST WE SHOW THAT IT IS POSSIBLE TO HAVE 5 CITIES, NO 3 OF WHICH ARE CONNECTED BY THE SAME METHOD OF TRANSPORTATION.

X ≡ BUS
O ≡ TRAIN



IF WE HAVE 6 CITIES, ONE CITY IS CONNECTED TO 5 OTHERS. OF THESE 5, AT LEAST 3 MUST BE CONNECTED TO THE ONE CITY BY THE SAME METHOD OF TRANSPORTATION, LET'S SAY BY BUS. IF ANY OF THESE 3 ARE CONNECTED BY BUS, WE HAVE THOSE 2 AND THE ORIGINAL ONE ALL CONNECTED BY THE SAME METHOD. THUS, NONE OF THESE 3 CAN BE CONNECTED BY BUS. THIS LEAVES THEM ALL CONNECTED BY TRAIN. HENCE, THERE IS NO WAY TO CONNECT $\boxed{6}$ CITIES SUCH THAT NO THREE ARE ALL DIRECTLY CONNECTED BY THE SAME METHOD OF TRANSPORTATION.

22) $P = \frac{1}{2}\left(\frac{4}{6}\right) + \frac{1}{2}\left(\frac{3}{3+x}\right) = \frac{1}{3} + \frac{3}{6+2x} = \frac{6+2x+9}{18+6x} = \frac{2x+15}{6x+18} \therefore$

$.5 < \frac{2x+15}{6x+18} < .6 \therefore 3x+9 < 2x+15 < 3.6x+10.8 \therefore 0 < 6-x < 0.6x+1.8 \therefore$

$6-x>0$ YIELDS $x<6 \therefore 6-x < .6x+1.8$ YIELDS $1.6x > 4.2 \therefore x > \frac{4.2}{1.6} \therefore x > 2\frac{5}{8} \therefore$

$x = 3, 4, 5$ ARE POSSIBLE $3+4+5 = \boxed{12}$

23) TOTAL # OF ARRANGEMENTS = (NUMBER OF WAYS TO PARTITION 5 OBJECTS INTO 3 GROUPS CONTAINING 2, 2 & 1 ELEMENTS) TIMES (NUMBER OF WAYS TO SELECT WHICH BOX HAS THE 1 ELEMENT) $= \left(\frac{5!}{2!\,2!\,1!}\right)\left(\binom{3}{1}\right) = (30)(3) = \boxed{90}$

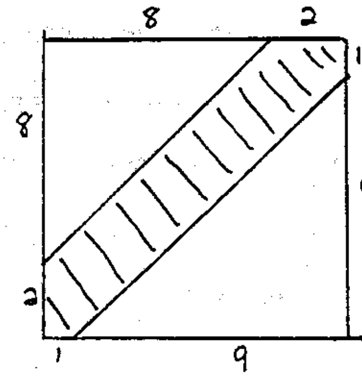
24) IF ANY SQUARE IS DIVIDED BY 12, THE REMAINDER CAN ONLY BE 0, 1, 4, OR 9 (PROVEN BY SQUARING $6k, 6k+1, \ldots, 6k+5$). THUS, THE SUM OF TWO SQUARES IS DIVISIBLE BY 12 IF AND ONLY IF BOTH THE SQUARES THEMSELVES ARE DIVISIBLE BY 12, BECAUSE THE ONLY COMBINATION OF TWO OF THE ABOVE REMAINDERS THAT YIELDS A TOTAL REMAINDER DIVISIBLE BY 12 IS 0+0. NOW, x^2 IS DIVISIBLE BY 12 IF AND ONLY IF x IS DIVISIBLE BY 6. HENCE, x AND y MUST BOTH BE MULTIPLES OF 6. HENCE,

$$P = \frac{(\text{\# WAYS TO PICK 2 MULTIPLES OF 6 LESS THAN 100})}{(\text{\# WAYS TO CHOOSE 2 NUMBERS LESS THAN 100})} = \frac{\binom{16}{2}}{\binom{100}{2}} = \frac{16(15)}{100(99)} = \boxed{\frac{4}{165}}$$

PROBABILITY & STATISTICS

(25) WE COUNT THE NUMBER OF SETS WHOSE DIFFERENCE BETWEEN THE GREATEST AND LEAST IS 4. THE LEAST NUMBER, X, MAY BE ANY NUMBER FROM 1 TO 6, THE GREATEST IS FIXED AT X+4, AND THE MIDDLE NUMBER MAY BE ANY OF THE THREE NUMBERS BETWEEN X & X+4. HENCE WE HAVE $6(1)(3) = 18$ SUCH SETS. THUS, OUR PROBABILITY IS $\frac{18}{\binom{10}{3}} = \frac{18}{120} = \boxed{\frac{3}{20}}$

(26) LET X BE THE MAN'S ARRIVAL TIME AND Y BE THE TAXI'S ARRIVAL. THUS, WE MUST HAVE $y-x<2$, IF $y>x$, AND $x-y<1$ IF $x>y$. THIS IS THE SHOWN REGION.



$$P = \frac{K_{SHADED}}{K_{SQUARE}} = \frac{K_{SQUARE} - K_{TRIANGLES}}{K_{SQUARE}} = \frac{10^2 - \frac{1}{2}(8^2+9^2)}{10^2} =$$

$$= \frac{100 - \frac{1}{2}(145)}{100} = \frac{27.5}{100} = \boxed{0.275}$$

(27) FIVE PEOPLE CAN CHOOSE FROM 15 CHAIRS IN $\binom{15}{5}$ WAYS. WE CAN MAKE A ONE-TO-ONE CORRELATION BETWEEN THE NUMBER OF WAYS FIVE PEOPLE CAN CHOOSE FROM 11 CHAIRS & THE NUMBER OF WAYS FIVE PEOPLE CAN CHOOSE FROM 15 CHAIRS WITH NO TWO PEOPLE ADJACENT. THIS CORRELATION CAN BE MADE BY NOTING THAT TO EACH CONFIGURATION OF 5 PEOPLE IN 11 CHAIRS ONE EMPTY CHAIR CAN BE PLACED BETWEEN EACH PAIR OF PEOPLE TO FORM A SET WITH 15 CHAIRS AND NO PEOPLE ADJACENT. FOR EXAMPLE, WITH O BEING EMPTY & X BEING FILLED: OXOOXXOXOOX BECOMES OXOOOXOXOOXOOOX. HENCE, WE HAVE $\binom{11}{5}$ WAYS FOR NO TWO TO BE ADJACENT.

THUS, $P = \frac{\binom{11}{5}}{\binom{15}{5}} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11} = \boxed{\frac{2}{13}}$

PROBABILITY & STATISTICS

(28) LET $f(n)$ = THE NUMBER OF WAYS TO GO FROM POINT n TO POINT 30. HENCE $f(n)=1$ FOR $25 \le n \le 30$ $\therefore$ $f(24)=f(25)+f(27)=2$ $\therefore$ $f(23)=f(24)+f(28)=3$. FOLLOWING THIS PATTERN, WE HAVE $f(5k)=f(5k+1)$, & for $1 \le a \le 4$, $k \le 4$: $f(5k+a)=f(5k+a+1)+f(5k+11-a)$:

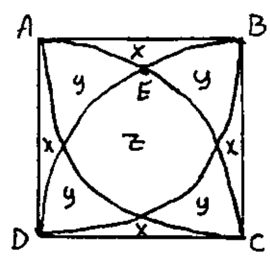
n	f(n)	n	f(n)	n	f(n)	n	f(n)	n	f(n)	n	f(n)
5	190	6	190	15	15	16	15	25	1	26	1
4	365	7	175	14	29	17	14	24	2	27	1
3	511	8	146	13	41	18	12	23	3	28	1
2	616	9	105	12	50	19	9	22	4	29	1
1	671	10	55	11	55	20	5	21	5	30	1

(29) LET $f(a,c)$ BE THE NUMBER OF SETS OF INTEGERS $\{a,b,c\}$ WITH $a<b<c$ SUCH THAT a, b, AND c CANNOT BE THE LENGTHS OF THE SIDES OF A TRIANGLE. HENCE, $f(a,c)=0$ FOR ALL $c<2a+1$. FOR ALL $a>19$, $f(a,c)=0$, BECAUSE $c \le 40$, WHICH IS LESS THAN $2a+1$ FOR ALL $a>19$. Now, $f(a,2a+k)=k$ (BECAUSE THE MIDDLE NUMBER CAN BE $a+1, a+2, \ldots$ OR $a+k$, A TOTAL OF k INTEGERS). THUS, THE TOTAL NUMBER OF SETS THAT CANNOT BE THE LENGTHS OF THE SIDES OF A TRIANGLE ARE: $\sum_{i=1}^{19} \sum_{j=2i+1}^{40} f(i,j) =$

$$= \sum_{i=1}^{19} \sum_{k=1}^{40-2i} f(i,2i+k) = \sum_{i=1}^{19} \sum_{k=1}^{40-2i} (k) = \sum_{i=1}^{19} \left(\frac{(40-2i)(41-2i)}{2} \right) = \sum_{i=1}^{19} (2i^2-81i+820) =$$

$$= 2\,\frac{(19)(20)(39)}{6} - 81\left(\frac{19(20)}{2}\right) + 820(19) = 270(19) \;\therefore\; P = 1 - \frac{270(19)}{\binom{40}{3}} = 1 - \frac{27}{52} = \boxed{\frac{25}{52}}$$

(30) IF WE DRAW UNIT CIRCLES AROUND EVERY LATTICE POINT, WE HAVE THE FOLLOWING UNIT SQUARE: THE CENTER REGION, AREA z, IS THE ONLY REGION WHICH IS IN THE CIRCLE AROUND EACH POINT. THIS, THEN, IS THE REGION WITHIN ONE UNIT OF 4 LATTICE POINTS.



$CE = DC = ED = 1$ (RADII OF UNIT CIRCLES) $\therefore$ $2x+y = K_{ABCD} - K_{\text{SECTOR } ABC} = 1 - \frac{\pi}{4}$

$2y+z = 2(K_{\text{SECTOR } ABC} - K_{\Delta ABC}) = 2(\pi/4 - 1/2) = \pi/2 - 1$ $\therefore$

$2y+z = 2(K_{\text{SECTOR } EDC} - K_{\Delta EDC}) + K_{\Delta EDC} = 2(\pi/6 - \sqrt{3}/4) + \frac{\sqrt{3}}{4} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$ $\therefore$

$x = (x+2y+z)-(2y+z) = 1 - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$ $\therefore$ $y = 1 - \pi/4 - 2x = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ $\therefore$ $z = \frac{\pi}{2} - 1 - 2y = 1 + \pi/3 - \sqrt{3}$

$$P = \frac{z}{K_{ABCD}} = \boxed{1 + \pi/3 - \sqrt{3}}$$