Gaither/Leto Invitational Calculus Solutions Jan 1992

D 1. L'Hôpital's Rule

$$\lim_{x \to 1/2^-} \frac{\frac{-8x}{2\sqrt{1-4x^2}}}{\frac{-2}{\sqrt{1-4x^2}}} = \lim_{x\to 1/2^-} (+2x) = 1$$

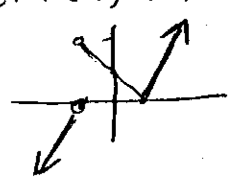
A 2. $f(x) = \dfrac{\frac{1}{2}+\frac{1}{2}\cos 2x}{\cos 2x}$

$$f\left(\tfrac{\pi}{12}\right) = \frac{\frac{1}{2}+\frac{1}{2}\cdot\frac{\sqrt3}{2}}{\frac{\sqrt3}{2}} = \frac{2+\sqrt3}{2\sqrt3} =$$

$$\frac{2\sqrt3+3}{6}$$

E 3. $f(x) = \dfrac{|x+1||x-1|}{x+1} = \begin{cases} |x-1| & \text{if } x>-1 \\ -|x-1| & \text{if } x<-1 \end{cases}$

Note: $f(-1)$ is not defined.

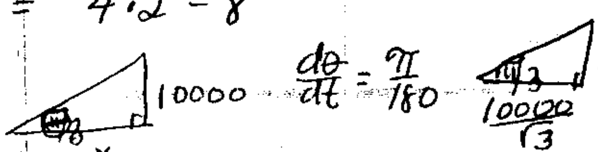


A 4. $f'(x) = 2\sec^2 2x \tan 2x \cdot 2$

$f'(\pi/8) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4}\cdot 2$

$= 4\cdot 2 = 8$

A 5.



$\dfrac{d\theta}{dt} = \dfrac{\pi}{180}$

$\tan\theta = \dfrac{10000}{x} \Rightarrow \sec^2\theta\dfrac{d\theta}{dt} = \dfrac{-10000}{x^2}$

$$\left(\sec^2\frac{\pi}{3}\right)\left(\frac{\pi}{180}\right) = -\frac{10000}{\left(\frac{10000}{\sqrt3}\right)^2}\frac{dx}{dt}$$

$$\left(\frac{10000}{-3}\right)\quad 4\cdot\frac{\pi}{180} = -\frac{3}{10000}\frac{dx}{dt}$$

$$\left|\frac{dx}{dt}\right| = \frac{2000\pi}{27}$$

C 6. $y^2 - 2y - 3 = 0$ when $x=1$

$(y-3)(y+1)=0$ $(1,3)$ and $(1,-1)$ are pts

$2y\frac{dy}{dx} - 2x\frac{dy}{dx} + y\cdot 2 = 3$

at $(1,3)$ $6\frac{dy}{dx} - 2\frac{dy}{dx} - 6 = 3$

$4\frac{dy}{dx} = 9$ $\frac{dy}{dx} = 9/4$

at $(1,-1)$ $-2\frac{dy}{dx} - 2\frac{dy}{dx} + 2 = 3$

$-4\frac{dy}{dx} = 1$ $\frac{dy}{dx} = -1/4$

Sum is 2

B 7. $a+3 = -3+b \Rightarrow a-b = -6$

$f'(x) = \begin{cases} 2ax & \text{if } x \ge 1 \\ -3 & \text{if } x<1 \end{cases} \Rightarrow 2a = -3$

$a = -3/2$

$-3/2 - b = -6$ $+b = +9/2$

D 8. $r(1+\sin\theta) = 4$

$r(1+\frac{y}{r}) = 4 \Rightarrow r+y = 4 \Rightarrow$

$r = 4-y \Rightarrow x^2+y^2 = 16-8y+y^2 \Rightarrow$

$x^2 = -8(y-2)$ $\quad y = -\frac{1}{8}x^2+2$

$y=0 \Rightarrow x = \pm 4$ $\quad \frac{dy}{dx} = -\frac{1}{4}x = 1$ at $x=-4$

C 9. $\frac{dy}{dx} = (\ln 2)\cdot 2^x$ and $(\ln 2)\cdot 2^x = 4 \Rightarrow$

$2^x = \frac{4}{\ln 2} \Rightarrow x = \dfrac{\ln\left(\frac{4}{\ln 2}\right)}{\ln 2} \Rightarrow x = \dfrac{\ln 4 - \ln(\ln 2)}{\ln 2}$

10. D $\quad f'(x) = xe^x + e^x \quad f''(x) = xe^x + 2e^x$

$e^x(x+1) = f'(x)$ $\quad e^x(x+2) = f''(x)$

$(-\infty,-1)$ | $-$ dec $\quad (-\infty,-2)$ | $-$ down

$(-1,\infty)$ | $+$ inc $\quad (-2,\infty)$ | $+$ up

$(-2,-1)$ is intersection

11. D $y = \frac{1}{2}\sin 2x\, e^x$

$\frac{dy}{dx} = \frac{1}{2}\sin 2x(e^x) + e^x\cos 2x$

$= \sin x\cos x\, e^x + (\cos 2x)e^x$

$= e^x(\sin x\cos x + \cos 2x)$

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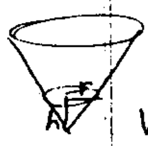
12. C $s(t)=t^3-6t^2+12t-9$
$v(t)=3t^2-12t+12$
$a(t)=6t-12$
$6t-12=0$ at $t=2$
$a(t)>0$ when $t>2$

13. B $f(x)=x^2+4x-6 \Rightarrow f'(x)=2x+4$
$f'(x)=2$ at $x=-1$ $f'(-1)=-9$

14. B  $\frac{dV}{dt}=-10$ $\pi r^2=9$ at $r=\frac{3}{\sqrt{\pi}}$

$\frac{8}{r}=\frac{12}{h}$ $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\cdot\frac{4h^2}{9}\cdot h$

$r=\frac{2h}{3}$ $V=\frac{4}{27}\pi h^3 \Rightarrow \frac{dV}{dt}=\frac{4}{9}\pi h^2\frac{dh}{dt}$

$\frac{3}{\sqrt{\pi}}=\frac{2h}{3}$

$h=\frac{9}{2\sqrt{\pi}}$ $-10=\frac{4}{9}\pi\cdot\frac{81}{4\pi}\cdot\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=-\frac{10}{9}$

15. B $f(x)=x^3-2x^2+x-2$ $f(x)=0$ at $x=2$ (others imag)
$f'(x)=3x^2-4x+1$
$f'(2)=5$ $y=5x-10$ is tangent line
or $5x-y-10=0$
$d=\frac{|2-10|}{\sqrt{26}}=\frac{4\sqrt{26}}{13}$

B 16. $x^2=\cos^2\theta$ and $y=2\cos^2\theta-1\Rightarrow$
$y=2x^2-1$
$\frac{dy}{dx}=4x$ and $\frac{d^2y}{dx^2}=4$

A 17. $2x-x\frac{dy}{dx}-y+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}(2y-x)=y-2x\Rightarrow\frac{dy}{dx}=\frac{y-2x}{2y-x}$

$\frac{d^2y}{dx^2}=\frac{(2y-x)(\frac{dy}{dx}-2)-(y-2x)(2\frac{dy}{dx}-1)}{(2y-x)^2}$

$\left(\frac{dy}{dx}-2=\frac{-3y}{2y-x}\right.$ and $\left.2\frac{dy}{dx}-1=\frac{-3x}{2y-x}\right)$

$\frac{d^2y}{dx^2}=\frac{(2y-x)(\frac{-3y}{2y-x})-(y-2x)(\frac{-3x}{2y-x})}{(2y-x)^2}$

$=\frac{3xy-6y^2+3xy-6x^2}{(2y-x)^3}$

See given equation

$=\frac{-6x^2+6xy-6y^2}{(2y-x)^3}=\frac{-6(x^2-xy+y^2)}{(2y-x)^3}=\frac{6(1)}{(x-2y)^3}$

E 18. $\int_5^3 f(x)dx=-4$ $\int_5^{10}=\int_5^3+\int_3^{10}=-4+14=10$

$3\int_5^{10}f(x)dx+\int_5^{10}2dx=3(10)+2x\Big|_5^{10}=$

$30+20-10=40$

A 19. $y=x^4-2x^2+1\Rightarrow y'=4x^3-4x\Rightarrow$
$y''=12x^2-4$ $y''=0\Rightarrow x=\pm\sqrt{1/3}$

$f'(\sqrt{1/3})=\frac{4}{3}\sqrt{1/3}-4\sqrt{1/3}=-\frac{8\sqrt3}{9}$ and $f'(-\sqrt{1/3})=\frac{8\sqrt3}{9}$

$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}=\frac{\left(-\frac{8\sqrt3}{9}-\frac{8\sqrt3}{9}\right)81}{\left(1+\frac{-64\cdot3}{81}\right)81}=\frac{-144\sqrt3}{-111}=\frac{48\sqrt3}{37}$

20. $y=x^5x^{-3}\cdot4x=4x^3$ $y'=12x^2$ A

21. x is # 3¢ increases $f(x)=(150+3x)(600-30x)$
$f'(x)=-180x-2700\Rightarrow f'(x)=0$ when $x=-15$
Subtract 15(.03) or \$.45. D

C 22. $g'(x)=5(h(x))^4\cdot h'(x)$
$g'(4)=5(-2)^4\cdot3=240$

A 23. $x+y=1$ $\frac{dy}{dx}=-1$

C 24. when $\frac{2x^2-x+3}{x^2-x+1}=2\Rightarrow -x+3=-2x+2$
or $x=-1$

C 25. $y=x^2+1$ $d^2=x^2+(x^2+1-2)^2$
(0,2) (x,x^2+1) $d^2=x^2+x^4-2x^2+1$
$d^2=x^4-x^2+1\Rightarrow (d^2)'=4x^3-2x$
$4x^3-2x=0\Rightarrow 2x(2x^2-1)=0\Rightarrow x=\sqrt{1/2}$

D 26. $\frac{dy}{dt}=900e^{3t/2}\Rightarrow y=600e^{3t/2}+C$
$2000=600+C\Rightarrow C=1400$
$y=600e^{3t/2}+1400\Rightarrow y=5200$ at $t=\ln 4$

D 27. $\int_0^{\ln 2}e^x dx=e^x\Big|_0^{\ln 2}=2-1=1$ $\frac{1}{2\ln 2}=\frac{1}{\ln 4}$

B 28. $g(x)=f(x^2)\Rightarrow g'(x)=f'(x^2)\cdot 2x$
$f'(x)=\sqrt{3x-2}\Rightarrow f'(4)=\sqrt{10}$
$g'(2)=f'(4)\cdot4=4\sqrt{10}$

E 29. $\lim_{h\to0}\frac{\int_1^{1+h}\sqrt{x^2+x}\,dx}{h}$ Use L'Hopital Rule

$=\lim_{h\to0}\frac{\sqrt{(1+h)^2+(1+h)}}{1}=\sqrt{2}$

C 30. $3x^2+2=\frac{72}{4}\Rightarrow 3x^2=16\Rightarrow$
$x=4\sqrt3/3$