Answer Key

1. B
2. A
3. B
4. C
5. B

6. C
7. A
8. D
9. B
10. C

11. E
12. C
13. A
14. D
15. B

16. D
17. A
18. D
19. B
20. A

21. C
22. B
23. A
24. B
25. C

26. E
27. D
28. A
29. B
30. C

Solutions:

1. $d = \sqrt{{r_1}^2 + {r_2}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$
   $d = \sqrt{\left(3\sqrt{3}\right)^2 + \left(\sqrt{5}\right)^2 - 2\left(3\sqrt{3}\right)\left(\sqrt{5}\right)\cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)}$
   $d = \sqrt{27 + 5 - 0} = \sqrt{32} = 4\sqrt{2}$
   Hence the answer is B

2. $56 = \pm\frac{1}{2}\begin{vmatrix} 3 & 3 & 1 \\ x & y & 1 \\ 6 & 2 & 1 \end{vmatrix}$
   $56 = \pm\frac{1}{2}\left(3\begin{vmatrix} y & 1 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} x & 1 \\ 6 & 1 \end{vmatrix} + \begin{vmatrix} x & y \\ 6 & 2 \end{vmatrix}\right)$
   $56 = \pm\frac{1}{2}[(3y - 6) - (3x - 18) + (2x - 6y)]$
   $56 = \pm\frac{1}{2}[-x - 3y + 12]$
   The ½ must be negative for *x* and *y* to be positive integers
   $-112 = -x - 3y + 12$
   $x = 124 - 3y$
   If *x* is a positive integer, there are 41 possible values for y.
   $y = \{1,2,3, \dots 41\}$
   Hence the answer is A

3. $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$
   There is one real solution and two imaginary solutions
   Hence the answer is B

4. $r = \frac{ep}{1 \pm e\sin(\theta)}$
   $r = \frac{6}{4 - 3\sin(\theta)} = \frac{3/2}{1 - (3/4)\sin(\theta)}$
   $e = 3/4$ The graph is an ellipse because $0 < e < 1$
   Hence the answer is C

5. $r = \frac{ep}{1 \pm e\sin(\theta)}$
   $r = \frac{6}{4 - 3\sin(\theta)} = \frac{3/2}{1 - (3/4)\sin(\theta)}$
   $ep = 3/2$
   $p = (3/2)(4/3) = 2$
   By definition, $r = \frac{ep}{1 \pm e\sin(\theta)}$ represents a conic section centered at $(0,0)$, with a major axis along the *y* direction. Also, by definition, due to the negative sign before the sine function, one directrix is below the minor axis. This directrix is $y = (0 - 2) = -2$.
   Hence the answer is B

6. $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$

$\sin\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \cos\left(\frac{5\pi}{12}\right)$

$\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) = \sin^2\left(\frac{\pi}{12}\right)$

$\cos(2\theta) = 1 - 2\sin^2(\theta)$

$\sin^2\left(\frac{\pi}{12}\right) = \left(-{}^{1}/_{2}\right)\left[\cos\left(\frac{\pi}{6}\right) - 1\right]$

$\sin^2\left(\frac{\pi}{12}\right) = \left(-{}^{1}/_{2}\right)\left[\frac{\sqrt{3}}{2} - 1\right] = \frac{2-\sqrt{3}}{4}$

Hence the answer is C

7. $(9)^2 + (12)^2 = (15)^2$ The triangle is a right triangle

$A = \frac{1}{2}(9)(12) = 54$

Shortest altitude is drawn from the longest side

$54 = \frac{1}{2}(h_{SHORTEST})(15) = \frac{36}{5}$

Hence the answer is A

8. Case I: The quantity of one to any power is one

$x^2 - 3x - 17 = 1$

$x^2 - 3x - 18 = 0$

$(x-6)(x+3) = 0 \rightarrow x = \{-3,6\}$

Case II: Any quantity to the zero power is one

$x^2 + x - 6 = 0$

$(x+3)(x-2) \rightarrow x = \{-3,2\}$

$x = \{-3,2,6\} \rightarrow$ The sum is 5

Hence the answer is D

9. $\frac{1}{z_A} = \frac{1}{4+8i}\left(\frac{4-8i}{4-8i}\right)$

$\frac{1}{z_A} = \frac{4-8i}{16+64} = \frac{1-2i}{20}$

$\left|\frac{1}{z_A}\right| = \sqrt{\frac{1}{400} + \frac{4}{400}} = \sqrt{\frac{1}{80}} = \frac{\sqrt{5}}{20}$

Hence the answer is B

10. $\cos(\theta) = \frac{u \bullet v}{\|u\|\|v\|}$

Direction vectors are $\langle 8, -4\rangle$ and $\langle 6,3\rangle$

$\cos(\theta) = \frac{(8)(6)+(-4)(3)}{(\sqrt{64+16})(\sqrt{36+9})} = \frac{48-12}{(\sqrt{80})(\sqrt{45})}$

$\cos(\theta) = \frac{36}{(4\sqrt{5})(3\sqrt{5})} = \frac{36}{12\times 5}$

$\theta = \arccos\left(\frac{36}{60}\right) = \arccos\left(\frac{3}{5}\right)$

Hence the answer is C

11. $[\sin(\theta) - \cos(\theta)]^2 = \frac{1}{4}$

$\sin^2(\theta) - 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = \frac{1}{4}$

$1 - \sin(2\theta) = \frac{1}{4} \rightarrow \sin(2\theta) = \frac{3}{4}$

$\cos(2\theta) = -\sqrt{1-\sin^2(2\theta)} = -\frac{\sqrt{7}}{16}$ because $\cos(2\theta)$ is in the second quadrant.

Hence the answer is E

12. $3120° - (8)(360°) = 240°$

Hence the answer is C

13. $\begin{vmatrix} \sin\left(\frac{n\pi}{4}\right) & \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(-\frac{n\pi}{4}\right) & \cos\left(-\frac{n\pi}{4}\right) \end{vmatrix} = \left[\sin\left(\frac{n\pi}{4}\right)\right]\left[\cos\left(-\frac{n\pi}{4}\right)\right] - \left[\cos\left(\frac{n\pi}{4}\right)\right]\left[\sin\left(-\frac{n\pi}{4}\right)\right]$

Cosine is an even function: $\cos\left(-\frac{n\pi}{4}\right) = \cos\left(\frac{n\pi}{4}\right)$

$\begin{vmatrix} \sin\left(\frac{n\pi}{4}\right) & \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(-\frac{n\pi}{4}\right) & \cos\left(-\frac{n\pi}{4}\right) \end{vmatrix} = \left[\sin\left(\frac{n\pi}{4}\right)\right]\left[\cos\left(\frac{n\pi}{4}\right)\right] - \left[\cos\left(\frac{n\pi}{4}\right)\right]\left[\sin\left(-\frac{n\pi}{4}\right)\right]$

Sine is an odd function: $\sin\left(\frac{n\pi}{4}\right) = -\sin\left(-\frac{n\pi}{4}\right)$

$\begin{vmatrix} \sin\left(\frac{n\pi}{4}\right) & \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(-\frac{n\pi}{4}\right) & \cos\left(-\frac{n\pi}{4}\right) \end{vmatrix} = \left[\sin\left(\frac{n\pi}{4}\right)\right]\left[\cos\left(\frac{n\pi}{4}\right)\right] + \left[\cos\left(\frac{n\pi}{4}\right)\right]\left[\sin\left(\frac{n\pi}{4}\right)\right]$

$\begin{vmatrix} \sin\left(\frac{n\pi}{4}\right) & \cos\left(\frac{n\pi}{4}\right) \\ \sin\left(-\frac{n\pi}{4}\right) & \cos\left(-\frac{n\pi}{4}\right) \end{vmatrix} = 2\left[\sin\left(\frac{n\pi}{4}\right)\right]\left[\cos\left(\frac{n\pi}{4}\right)\right] = \sin\left(\frac{n\pi}{2}\right)$

$$\sum_{n=1}^{8}\left[\sin\left(\frac{n\pi}{2}\right)\right] = 1 + 0 - 1 + 0 + 1 + 0 - 1 + 0 = 0$$

Hence the answer is A

14. $f(x) = 2 - 2\cos^2(\theta)$

$f(x) = 1 + [1 - 2\cos^2(\theta)] = 1 - \cos(2\theta)$

$p = \frac{2\pi}{|2|} = \pi$

Hence the answer is D

15. $\begin{bmatrix} 28 & -16 \\ 8 & -4 \end{bmatrix}^{-1} = \frac{1}{-112-(-128)}\begin{bmatrix} -4 & 16 \\ -8 & 28 \end{bmatrix}$

$\begin{bmatrix} 28 & -16 \\ 8 & -4 \end{bmatrix}^{-1} = \frac{1}{16}\begin{bmatrix} -4 & 16 \\ -8 & 28 \end{bmatrix} = \begin{bmatrix} -1/4 & 1 \\ -1/2 & 7/4 \end{bmatrix}$

Hence the answer is B

16. $\cos\left(\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

Hence the answer is D

17. $Number\ of\ Petals = 2020 \times 2 = 4040$

Hence the answer is A

18. $\left(\sqrt{6}-i\sqrt{2}\right)^8 = \left(2\sqrt{2}\right)^8\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^8$

$\left(\sqrt{6}-i\sqrt{2}\right)^8 = (2)^8\left(\sqrt{2}\right)^8\left(cis\left(\frac{11\pi}{6}\right)\right)^8$

$\left(\sqrt{6}-i\sqrt{2}\right)^8 = (2)^{12}\left(cis\left(\frac{88\pi}{6}\right)\right)$

$\left(\sqrt{6}-i\sqrt{2}\right)^8 = (2)^{12}\left(cis\left(\frac{4\pi}{6}\right)\right)$

$\left(\sqrt{6}-i\sqrt{2}\right)^8 = (2)^{12}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$

$\left(\sqrt{6}-i\sqrt{2}\right)^8 = (2)^{11}\left(-1+i\sqrt{3}\right) = -2048+2048\sqrt{3}i$

Hence the answer is D

19. $2\sin(x)+\cos(2x) = 2\sin(x)+1-2\sin^2(x)$

Analogous to a downward facing parabola ax^2+bx+c

Maximum occurs at the vertex, within each cycle of 2π

$\sin(x) = -\frac{b}{2a} = -\frac{-2}{2(-2)} = \frac{1}{2}$

The maximum value is $2\left(\frac{1}{2}\right)+1-2\left(\frac{1}{2}\right)^2 = \frac{3}{2}$

Hence the answer B

20. $|z+3| = 2|z-2i|$

$|z| = \sqrt{x^2+y^2}$

$\sqrt{(x+3)^2+y^2} = 2\sqrt{x^2+(y-2)^2}$

$(x+3)^2+y^2 = 4[x^2+(y-2)^2]$

$x^2+6x+9+y^2 = 4x^2+4y^2-16y+16$

$-7 = 3x^2-6x+3y^2-16y$

Both coefficients for the x^2 and the y^2 term are the same, so this is a circle

$-7 = 3(x^2-2x)+3\left(y^2-\frac{16}{3}y\right)$

$3(x^2-2x+1)+3\left(y^2-\frac{16}{3}y+\frac{64}{9}\right) = -7+3+\frac{64}{3}$

$3(x-1)^2+3\left(y-\frac{8}{3}\right)^2 = \frac{52}{3}$

$(x-1)^2+\left(y-\frac{8}{3}\right)^2 = \frac{52}{9}$

This is a circle with $A = \frac{52}{9}\pi$

Hence the answer is A

21. $p = \frac{2\pi}{B}$

$\frac{\pi}{8} = \frac{2\pi}{B} \rightarrow B = 16$

$14 = \text{Acos}\left(16\left(\frac{\pi}{3}\right)+\frac{\pi}{6}\right)+D$

$14 = \text{Acos}\left(\frac{16\pi}{3}+\frac{\pi}{6}\right)+D$

$14 = \text{Acos}\left(\frac{4\pi}{3}+\frac{\pi}{6}\right)+D \quad \rightarrow \quad D = 14$

The amplitude is half the distance from the top of the graph to the bottom of the graph

$A = 8$ or $A = -8$

$|AD| = 112$

Hence the answer is C

22. $\log_{49}(343) + \log_{49}(7) - \log_2 512 = \frac{3\log 7}{2\log 7} + \frac{\log 7}{2\log 7} - 9$

$\log_{49}(343) + \log_{49}(7) - \log_2 512 = \frac{3}{2} + \frac{1}{2} - 9 = -7$

Hence the answer is B

23. $z_1 z_2 = (2-4i)(4+2i)$

$z_1 z_2 = 8 + 4i - 16i + 8$

$z_1 z_2 = 16 - 12i$

$\arg(z_1 z_2) = \arctan\left(-\frac{3}{4}\right)$

Hence the answer is A

24. $\frac{z_2}{z_1} = \frac{4+2i}{2-4i}\left(\frac{2+4i}{2+4i}\right) = \frac{8+16i+4i-8}{4+16} = i$

$r = \sqrt{0+1} = 1$

$\frac{z_2}{z_1} = cis\left(\frac{\pi}{2}\right)$

Hence the answer is B

25. The first inequality is positive when $x > -3$

The second inequality is positive when $x > 1$

Case 1 on interval $(-\infty, -3)$: $(3-x) - (1-x) > 1$

$2 > 1$ Infinitely many solutions

Case 2 on interval $(-3,1)$: $(x+3) - (1-x) > 1$

$2x + 2 > 1$

$x > -\frac{1}{2}$

Case 3 on interval $(1, \infty)$: $(x+3) - (x-1) > 1$

$4 > 1$ Infinitely many solutions

Hence the answer is C

26. ~~Option I: $\langle 6,3,3\rangle \bullet \langle 5,-6,-4\rangle = 30 - 18 - 12 = 0$~~

~~Option II: $\langle 4,8,-7\rangle \bullet \langle 5,-6,-4\rangle = 20 - 48 + 28 = 0$~~

~~Option III: $\langle -2,5,-4\rangle \bullet \langle 5,-6,-4\rangle = -10 - 30 + 16 = -24$~~

~~**Hence the answer A**~~

CORRECTION: The vector of coefficients of a plane is orthogonal to the plane, and vectors perpendicular to it are not perpendicular to the plane itself.

Hence the answer is E

27. $\frac{2^6-3^6}{2^3+3^3} = \frac{(2^3-3^3)(2^3+3^3)}{2^3+3^3}$

$\frac{6^6-4^6}{2^3+3^3} = (2^3 - 3^3) = -19$

Hence the answer is D

28. $f(x) = x^3 - 3x^2 + 5x - 15$
    $f(x) = x(x^2 + 5) - 3(x^2 + 5) = (x - 3)(x^2 + 5)$
    $x = \{\pm 5i, 3\}$
    The sum of the imaginary roots is 0
    Hence the answer is A

29. $B^2 - 4AC = 16 - 4(5)(-3) = 76$
    $B^2 - 4AC > 0$ The conic is a Hyperbola
    Hence the answer is B

30. $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x & 2 \\ y & 5 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 26 & 24 \end{bmatrix}$
    $\begin{bmatrix} 3x - y & 1 \\ 2x + 4y & 24 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 26 & 24 \end{bmatrix}$
    $y = 3x - 11$
    $2x + 4(3x - 11) = 26$
    $14x = 70$
    $x = 5$
    $y = 3(5) - 11 = 4$
    $(x - y) = 1$
    Hence the answer is C