Answer Key

1. C
2. C
3. C
4. A
5. A
6. A
7. C
8. B
9. C
10. C
11. B
12. C
13. D
14. A
15. C
16. B
17. D
18. C
19. B
20. C
21. B
22. C
23. D
24. E
25. D
26. B
27. C
28. B
29. B
30. B

Solutions

1. Three points determine a unique plane. **C.**

2. The contrapositive of "If Buffy eats a banana on a given day, then it will necessarily rain during both of the following two days" is "If it does not rain on either of the two days following the given day, then Buffy did not eat a banana on the given day". This means (I) and (II) are true, but we know nothing about (III): Buffy eating a banana on a given day does not necessarily mean it will rain that day. **C.**

3. By definition, $AC = 32$, so $AD = 16$. This means $BD = 64 - 16 = 48$, so $BE = 24$. This means $AE = 64 - 24 = \mathbf{40}$. **C.**

4. Since $\angle BPD$ and $\angle APC$ are vertical angles, they have the same measure. This means $3x - 5 = 2x + 31$, or $x = 36$. Then $\angle PCA = 144° - 90° = 54°$. Since the angles of a triangle add up to $180°$, this makes $\angle BAC = 180° - 103° - 54° = 23°$.**A.**

5. Let x and y be the measures(in degrees) of angles X and Y, respectively. The information given implies $x + y = 90$ and $180 - (90 - x) = 3(90 - y)$. Substituting and solving gives $x = y = 45$, so the difference between x and y is 0.**A.**

6. Since the angles of a triangle add up to $180°$, we must have $\angle BAC = 30°$, $\angle ABC = 60°$, and $\angle ACB = 90°$. Since ABC is a $30° - 60° - 90°$ triangle with smallest side length 1, the length of the other leg is $\sqrt{3}$ and thus ABC has area $\frac{\sqrt{3}}{2}$.**A.**

7. Since the angles are in arithmetic progression and sum to $540°$, the middle term must be $108°$. If d is the common difference of the arithmetic progression, then the sum we want is $(108 + 2d)° + (108 - 2d)° = 216°$.**C.**

8. Let $\angle BAC = x°$. Since $AD = BD$, we have $\angle ABD = x°$ and $\angle ADB = (180 - 2x)°$. Since $BD = CB$, we have $\angle BDC = \angle ACB = 2x°$. On one hand, $\angle CBD = (180 - 4x)°$ by observing triangle DBC, but it's also equal to $(2x - x)° = x°$. Then $180 - 4x = x$, so $x = 36$.**B.**

9. There are only 7 unordered triples: $(2, 3, 4)$, $(2, 4, 5)$, $(2, 5, 6)$, $(3, 4, 5)$, $(3, 4, 6)$, $(3, 5, 6)$, and $(4, 5, 6)$. **C.**

10. Since $33^2 + 56^2 = 1089 + 3136 = 4225 < 4356 = 66^2$, the triangle is obtuse. **C.**

11. A cube has 12 edges. For any fixed edge, there are 3 other edges that are parallel to it. This means there are $12 \cdot 3 = 36$ ordered pairs, but this counts each pair of edges twice. The answer is then $\frac{36}{2} = 18$.**B.**

12. Like the above question, for a fixed edge, there are 4 edges that are skew to it. This means there are $\frac{1}{2} \cdot 4 \cdot 12 = 24$ unordered pairs of skewed edges. **C.**

13. The equation of the line passing through $(3, 4)$ and $(5, 12)$ is $y = 4x - 8$. Since $(a, 3a + 1)$ lies on this line, we must have $3a + 1 = 4a - 8$, or $a = 9$.**D.**

14. By the distance formula, we have $(x + 3)^2 + (x + 7)^2 = \left(2\sqrt{2}\right)^2$, or $2x^2 + 20x + 50 = 2(x + 5)^2 = 0$, so the only solution is $x = -5$.**A.**

15. Let $x = 2$ intersect the circle at distinct points A and B, let O be the origin, and let M be the midpoint of segment AB. Since OAM is a right triangle and $OA = 2OM$, it is a $30° - 60° - 90°$ right triangle. This means the smaller sector of the circle cut out by radii OA and OB has central angle $120°$, so it has area $\frac{16\pi}{3}$. Subtracting the area of triangle OAB gives the area we want, so the answer is $\frac{16\pi}{3} - 4\sqrt{3}$.**C.**

16. This means $\frac{360}{x} = 10x$, or $x = 6$. A polygon with $10x = 60$sides has $\frac{60(60-3)}{2} = 1710$diagonals. **B.**

17. It's well-known that $D(n) = \frac{n(n-3)}{2}$ and $S(n) = 180(n-2)$. This means Mr. Pickle's winnings (in cents) are $180(n-2) - 12n(n-3) = -12n^2 + 216n - 360 = -12(n-9)^2 + 612$, which is maximized at $n = 9$.**D.**

18. Let $\angle ABC = x°$. Since $AD = BD$, we have $\angle DAB = x°$, meaning $\angle BAC = 2x°$ because it's bisected by AD. This means $3x + 48 = 180$, so $x = 44$. We want $\angle BAC = 2x° = 88°$. **C.**

19. Since $AB = MC$ and $AB \parallel MC$, it follows that $AMCB$ is a parallelogram. This means $AM = BC = 7$, and similarly, $AD = BM = 5$. Since $AB = \frac{12}{2} = 6$, the perimeter of ABM is just $5 + 6 + 7 = 18$.**B.**

20. I and III are necessarily true because they are corresponding lengths in similar triangles, i.e. they are one-dimensional and thus the scale factor of $1:2$ is raised to the first power. II is not necessarily true because it's only guaranteed that $AB:PQ = 1:2$; AB and PR are not corresponding sides. IV is always false: since area is a two-dimensional quantity, so the ratio is always $1^2:2^2$.**C.**

21. Observe that there are two possible choices for the location of E: in the interior of $ABCD$ and outside $ABCD$.Regardless, $CDE$is isosceles with base $CD = 4$,so we only need to find the two different distances from E to CD.If Elies in the interior of $ABCD$, then the distance from E to $CD$is $4 - 2\sqrt{3}$. If E lies outside $ABCD$, then the distance from $E$to $CD$is $4 + 2\sqrt{3}$instead. This means the product of the areas is $\frac{1}{2} \cdot 4 \cdot (4 - 2\sqrt{3}) \cdot \frac{1}{2} \cdot 4 \cdot (4 + 2\sqrt{3}) = 16$.**B.**

22. The region contained in both triangle $ACE$and square $ABCD$is the interior of either triangle ABCor triangle, both of which have area $\frac{1}{2} \cdot 4 \cdot 4 = 8$.Since ACEhas side length $AC = 4\sqrt{2}$, the area of $ACE$is $\frac{32\sqrt{3}}{4} = 8\sqrt{3}$.Their difference is precisely what we want, and is just $8\sqrt{3} - 8$.**C.**

23. We can compute the area of the kite by finding the areas of triangles ABD and BCD.Since $ABD$is equilateral with side length 4, it has area $4\sqrt{3}$.Let M be the midpoint of segment BD.Since triangle $CBD$is isosceles, we have $CM = \sqrt{14^2 - 2^2} = 8\sqrt{3}$, so the area of triangle $CBD$is $\frac{1}{2} \cdot 4 \cdot 8\sqrt{3} = 16\sqrt{3}$. The area of the entire kite is then $4\sqrt{3} + 16\sqrt{3} = 20\sqrt{3}$.**D.**

24. Observe that $APBQ$is actually a (possibly concave) kite: this is because $AP = AQ$ and $BP = BQ$.Moreover, since $\angle APB = \angle AQB$, it follows that $\angle APB = \angle AQB = 90°$because they are opposite angles of a cyclic quadrilateral and thus must sum to $180°$.Then by Pythagoras, we have $BP = BQ = \sqrt{11^2 - 5^2} = 4\sqrt{6}$,so the area of $APBQ$is $2 \cdot \frac{1}{2} \cdot 5 \cdot 4\sqrt{6} = 20\sqrt{6}$.**E.**

25. By Heron's formula, the area of ABC is $\sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84$. This means $84 = \frac{1}{2} \cdot BC \cdot AD = 7AD$, so $AD = 12$. **D.**

26. Since $AD = 12$, we have $BD = \sqrt{13^2 - 12^2} = 5$. Because E is the midpoint of segment BC, it follows that $BE = 7$, so $DE = 7 - 5 = 2$. By Pythagoras, $AE = \sqrt{AD^2 + DE^2} = \sqrt{12^2 + 2^2} = 2\sqrt{37}$. **B.**

27. Using a similar idea to the previous problem, we just need to find DF. By the Angle Bisector Theorem, we have $BF : CF = 13 : 15$ and $BF + CF = 14$, so $BF = \frac{13}{2}$. This means $DF = BF - BD = \frac{13}{2} - 5 = \frac{3}{2}$, so $AF = \sqrt{AD + DF^2} = \sqrt{12^2 + (\frac{3}{2})^2} = \frac{3\sqrt{65}}{2}$. **C.**

28. Let a be the length of segment BC. By the Angle Bisector Theorem, we have $BD = \frac{5}{9}a$ and since $BD : CD = 5 : 4$. The main idea is to observe that $\angle DBA = \angle BAD = \angle DAC$ since BAD is isosceles and AD is an angle bisector. This means $DAC \sim ABC$ by AA similarity, as $\angle ACD = \angle ACB$ and $\angle DAC = CBA$. Then $\frac{AC}{DC} = \frac{BC}{AC}$ by using the corresponding sides of the similar triangles. This means $AC^2 = CD \cdot CB$, or $16 = \frac{4}{9}a \cdot a = \frac{4}{9}a^2$. Solving easily gives $a = 6$. **B.**

29. Let A' be the reflection of A about line DC and let P' be the intersection of MA' and DC. We'll show that P' is the point we want. Pick a point Q on line DC. By the Triangle Inequality, it follows that $AQ + QM = A'Q + QM \leq AM$, with equality when Q lies on $A'M$. This means $P' = P$ is the point we want, as AM is constant regardless of the choice of the point on CD. To find the area, we'll subtract out the areas of right triangles ABM, MCP, and ADP from the area of $ABCD$. Note that by similar triangles $A'DP$ and MCP, we have $DP : DC = A'D : MC = 2 : 1$, so $DP = 10$ and $CP = 5$. Simple calculation gives the area of triangle AMP as $240 - 60 - 80 - 20 = 80$. **B.**

30. Since triangles DAC and ABC are similar, we have $\angle DAC = \angle ABC = \angle ACB$, so lines AD and BC are parallel. Moreover, since $AC = 4$, this means $AD = 2 \cdot 4 = 8$. Let E be the foot of the altitude from B onto line AD. Then $AE = \frac{1}{2}BC = 1$, so $BE = \sqrt{4^2 - 1^2} = \sqrt{15}$. Finally, $DE = AE + AD = 9$, so $BD = \sqrt{9^2 + 15} = 4\sqrt{6}$. **B.**