2022 January Statewide Calculus Individual Solutions

Answer Key:
1. B
2. A
3. E
4. C
5. C
6. E
7. A
8. B
9. A
10. E
11. B
12. C
13. A
14. A
15. D
16. B
17. D
18. A
19. D
20. B
21. E
22. D
23. A
24. A
25. C
26. C
27. B
28. D
29. A
30. B

1. Using Power Rule, $f'(2) = 5(2^4) - 10(4)(2^3) + 40(3)(2^2) - 80(2)(2) + 80 = 0$. Alternatively, note that $f(x) = (x-2)^5$, so $f'(x) = 5(x-2)^4 \rightarrow f'(2) = 5(2-2)^4 = 0$. **B.**

2. The marginal enjoyments that Eric and Eddie get are $R'(s)$ and $D'(s)$, respectively. Therefore, we are looking for the s at which $R'(s) = D'(s)$. $R'(s) = -8s + 64$. $D'(s) = 2s$. $-8s + 64 = 2s$. $10s = 64$. $s = 6.4$. **A.**

3. The limit as x approaches 2 from the left does not exist because the quantity under the radical would be negative. Therefore, the limit as x approaches 2 does not exist. **E**

4. The function $f(x) = x^t e^x$ has second derivative $f''(x) = e^x + tx^{t-1}e^x + tx^{t-1}e^x + t(t-1)x^{t-2}e^x = e^x x^{t-2}(x^2 + 2tx + t^2 - t)$. Since $t < 1$, the x-coordinates of the two inflection points are the solutions to $x^2 + 2tx + t^2 - t = 0$. The positive solution is easily calculated to be $\frac{-2t+\sqrt{4t^2-4(t^2-t)}}{2} = \sqrt{t} - t = -\left(\frac{1}{2} - \sqrt{t}\right)^2 + \frac{1}{4} \leq \frac{1}{4}$ with equality at $t = \frac{1}{2}$. **C.**

5. The line tangent to f at $x = 2$ has slope $f'(2) = 4 + 4 = 8$ and passes through $\left(2, f(2)\right) = (2, 15)$. This means the tangent line has equation $y = 8x - 1$. Similarly, the line tangent to g at $x = 2$ has slope $g'(2) = 4 - 2 = 2$ and passes through $\left(2, g(2)\right) = (2, 5)$, so this tangent line has equation $y = 2x + 1$. The two lines intersect at $\left(\frac{1}{3}, \frac{5}{3}\right)$, so $a + b = 2$. **C.**

6. $g'(x) = \frac{1}{f'(g(x))}$. To find $g(\frac{15}{4})$ we solve $\frac{15}{4} = 2y^3 + 3y + 2$, which gives $y = \frac{1}{2}$. $f'\left(\frac{1}{2}\right) = 2(3)\left(\frac{1}{2}\right)^2 + 3 = \frac{9}{2}$. Therefore, $g'(x) = \frac{1}{\frac{9}{2}} = \frac{2}{9}$. Since we are looking for the slope of the normal line and $\frac{9}{2}$ is the slope of the tangent line, our answer is the negative reciprocal of $\frac{9}{2}$, which is $-\frac{2}{9}$. **E.**

7. The volume of the balloon can be modeled by $V = \frac{1}{2} * \frac{4}{3}\pi r^3 + \frac{1}{2} * \frac{4}{3}\pi r^3 + 20r(\pi r^2) = \frac{64}{3}\pi r^3$. Differentiating with respect to time, $\frac{dV}{dt} = 64\pi r^2 \frac{dr}{dt}$. Plugging in our values of $r = 0.5$ and $\frac{dV}{dt} = 32$, $32 = 64\pi(0.5)^2 \frac{dr}{dt}$. $\frac{dr}{dt} = \frac{2}{\pi}$. **A.**

8. Our approximation is $f\left(\frac{1}{3}\right) \approx f\left(\frac{2}{3}\right) + f'\left(\frac{2}{3}\right)(-\frac{1}{3})$. We can find $f'\left(\frac{2}{3}\right)$ using $f'(\frac{2}{3}) = (\frac{2}{3})^2 f\left(\frac{2}{3}\right) + 2 = (\frac{2}{3})^2 9 + 2 = 6$. $f\left(\frac{1}{3}\right) \approx 9 + 6\left(-\frac{1}{3}\right) = 7$. **B.**

9. Note: in this solution, $\angle X$ is used to represent $m\angle X$ for simplicity. We will examine the triangle formed by Andrew (vertex A), Bailey (vertex B), and the center

of the merry-go-round (vertex C). We are looking for the absolute value of the rate of change in $\angle A$ with respect to time, which is $\frac{d\angle A}{dt}$. We can use Law of Sines: $\frac{BC}{sin(\angle A)} = \frac{AC}{sin(\angle B)} \rightarrow \frac{6\sqrt{2}}{sin(\angle A)} = \frac{6\sqrt{2}+(12-6\sqrt{2})}{sin(\angle B)} \rightarrow 6\sqrt{2}sin(\angle B) = 12sin(\angle A) \rightarrow sin(\angle B) = \sqrt{2}sin(\angle A)$. Differentiating both sides of our equation with respect to time, we get: $cos(\angle B)\frac{d\angle B}{dt} = \sqrt{2}cos(\angle A)\frac{d\angle A}{dt}$. Now we need to find and plug in $\angle B$, $\frac{d\angle B}{dt}$, and $\angle A$. One second after Bailey passes the point closest to Andrew, $\angle C$ is $0 + 1 * \frac{7\pi}{12} = \frac{7\pi}{12}$. If drop the altitude from C to $\overline{AB}$, we can find that $\angle A = \frac{\pi}{6}$ and $\angle B = \frac{\pi}{4}$. Since $\angle A + \angle B + \angle C = \pi$, $\frac{d\angle A}{dt} + \frac{d\angle B}{dt} + \frac{d\angle C}{dt} = 0$. We can substitute in $\frac{7\pi}{12}$ for $\frac{d\angle C}{dt}$, giving $\frac{d\angle B}{dt} = -\frac{d\angle A}{dt} - \frac{7\pi}{12}$. Finally, when we substitute in these values of $\angle B$, $\frac{d\angle B}{dt}$, and $\angle A$, we get: $cos\left(\frac{\pi}{4}\right)(-\frac{d\angle A}{dt} - \frac{7\pi}{12}) = \sqrt{2}cos\left(\frac{\pi}{6}\right)\frac{d\angle A}{dt}$. Solving for $\frac{d\angle A}{dt}$, we find $\frac{d\angle A}{dt} = \frac{7\pi-7\pi\sqrt{3}}{24}$. $\left|\frac{7\pi-7\pi\sqrt{3}}{24}\right| = \frac{7\pi\sqrt{3}-7\pi}{24}$. $a + b + c + d = 7 + 3 + 7 + 24 = 41$. **A.**

10. The Mean Value Theorem for Derivatives guarantees at least one value of x in the interval such that $f'(x)$ is equal to the slop of the secant line between the endpoints of the interval. That slope is: $\frac{f(3)-f(-2)}{3-(-2)} = \frac{38-18}{5} = \frac{20}{5} = 4$. At this point, we could solve for the least value that the theorem guarantees (which is $x = \frac{1}{2} - \frac{\sqrt{3}}{2}$) and plug it in to get our final answer. However, the intended solution is to notice that the expression that the question asks for is just $-\frac{3}{2}$ times the difference quotient. Since the limit of the difference quotient is the slope of $f(x)$ at the point we are looking for and we know this value to be 4, our answer is $-\frac{3}{2}(4) = -6$. **E.**

11. Condition I: $f(x)$ is decreasing when $f'(x) < 0$, which in this case means when $-\sin(x) < 0$. This is satisfied on the interval $(0, \pi)$. Condition II: $f(x)$ is concave up when $f''(x) > 0$, which in this case means when $-\cos(x) > 0$. This is satisfied on the interval $(-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Condition III: $f(x) < f'(2x) \rightarrow \cos(x) < -2\sin(2x) \rightarrow \cos(x) < -2\cos(x)\sin(x) \rightarrow \cos(x)(1 + 2\sin(x)) < 0$. This is satisfied on the interval $(-\pi, -\frac{5\pi}{6}) \cup (-\frac{\pi}{2}, -\frac{\pi}{6}) \cup (\frac{\pi}{2}, \pi)$. Conditions I and II only (meaning and not Condition III) are satisfied on no interval. Conditions I and III only are satisfied on no interval. Conditions II and III only are satisfied on the interval $\left(-\pi, -\frac{5\pi}{6}\right)$. The combined length of these intervals is $\frac{\pi}{6}$, and the length of the interval $(-\pi, \pi)$ is 2π. $\frac{\frac{\pi}{6}}{2\pi} = \frac{1}{12}$. **B.**

12. The first precondition is necessary for all four theorems ($q = 4$). The second precondition is not necessary for any of the four theorems because the function need not be differentiable at the endpoints on the interval ($r = 0$). The third precondition is necessary for the Mean Value Theorem for Derivatives and Rolle's

Theorem ($s = 2$). The fourth precondition is necessary for Rolle's Theorem only ($t = 1$). $q + r + s + t = 4 + 0 + 2 + 1 = 7$. **C.**

13. $\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\left(1 - \frac{1}{2n}\right)^{3n} = \lim_{n\to\infty}\left(1 - \frac{1}{2n}\right)^{(\frac{3}{2})(2n)}$. If we make the substitution $m = 2n$, our expression becomes: $\lim_{m\to\infty}\left(1 - \frac{1}{m}\right)^{(\frac{3}{2})(m)} = [\lim_{m\to\infty}\left(1 - \frac{1}{m}\right)^{m}]^{\frac{3}{2}} = (e^{-1})^{\frac{3}{2}} = e^{-\frac{3}{2}}$. Therefore, $\lim_{n\to\infty} ln(f(n)) = ln(\lim_{n\to\infty} f(n)) = ln\left(e^{-\frac{3}{2}}\right) = -\frac{3}{2}$. **A.**

14. f is actually continuous at $x = 0$! This is because for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| = |x| < \epsilon$; just take $\delta = \frac{\epsilon}{2}$. Intuitively, this means that as x gets arbitrarily close to zero, so does $f(x)$. It's also easy to see that f is not continuous at any other point because for every rational, we can find an irrational arbitrarily close to it and vice verse. **A.**

15. Since $P(2) = P'(2) = 0$, it follows that $(x-2)^2$ divides $P(x)$ by the Product Rule. Let $P(x) = (x-2)^2(x-a)$. Since $P(1) = 5$, it follows that $a = -4$. This means $P(x) = (x-2)^2(x-4) = x^3 - 12x + 16$, which has its only inflection point at $(0, 16)$. Thus $a + b = 16$. **D.**

16. Implicitly differentiating, we get $\frac{2xx'}{4} + \frac{2yy'}{9} = 0$, or $\frac{y'}{x'} = -\frac{9x}{4y}$. Plugging in $(x, y) = \left(1, \frac{3\sqrt{3}}{2}\right)$ gives $\frac{dy}{dx} = -\frac{\sqrt{3}}{2}$. **B.**

17. Konwoo's velocity at time t is given by $x'(t) = 3t^2 - 18t + 24 = 3(t-2)(t-4)$ and his acceleration at time t is given by $x''(t) = 6t - 18 = 6(t-3)$. Konwoo's speed is increasing when both acceleration and velocity have the same sign. Konwoo's velocity is negative on $(2, 4)$ and his acceleration is negative on $(0, 3)$. Konwoo's velocity is positive on $(0, 2) \cup (4, 6)$ and his acceleration is positive on $(3, 6)$. This means his speed increases on the intervals $(2, 3) \cup (4, 6)$. **D.**

18. We will call x the distance from the center axis of the Bundt cake and semi-sphere to the inner face of the cake, otherwise known as the inner radius of the cake. The volume of the cake can be expressed as: $V(x) = \pi[(outer\ radius)^2 - (inner\ radius)^2](height) = \pi[(x+4)^2 - x^2](height) = \pi(8x + 16)(height)$. We can solve for the height using the Pythagorean Theorem: $(2\sqrt{10})^2 = (x+4)^2 + height^2 \rightarrow height = \sqrt{-x^2 - 8x + 24}$. Substituting this into our equation for volume, we get: $V(x) = \pi(8x + 16)\left(\sqrt{-x^2 - 8x + 24}\right) = 8\pi(x + 2)\left(\sqrt{-x^2 - 8x + 24}\right)$ for $0 \le x \le 2\sqrt{10} - 4$. Now, we maximize this expression. $V'(x) = 8\pi\left[(x+2)\left(\frac{1}{2} * \frac{1}{\sqrt{-x^2-8x+24}} * (-2x - 8) + \left(\sqrt{-x^2 - 8x + 24}\right)(1)\right)\right] = 8\pi\frac{(-x^2-6x-8)+(-x^2-8x+24)}{\sqrt{-x^2-8x+24}} = 8\pi\frac{-2x^2-14x+16}{\sqrt{-x^2-8x+24}} = 16\pi\frac{-x^2-7x+8}{\sqrt{-x^2-8x+24}}$. Therefore, the points we must check are: our critical points at $x = 1$ and $x = 2\sqrt{10} - 4$ ($x = -8$ and

$-2\sqrt{10}-4$ are not in our domain) and our additional endpoint at $x=0$. When we use our equation for volume, we get: at $x=1, V=24\pi\sqrt{15}$; at $x=2\sqrt{10}-4, V=0$; and at $x=0, V=32\pi\sqrt{6}$. The maximum of these three volumes is $V=24\pi\sqrt{15}$, so that is our answer. **A.**

19. Taking the derivative with respect to x yields $\frac{dy}{dx}=2^x\ln(2)+x\frac{dy}{dx}+y$. (solving for $\frac{dy}{dx}$ yields $\frac{dy}{dx}=\frac{2^x\ln(2)+y}{1-x}$). Taking the derivative with respect to x again yields $\frac{d^2y}{dx^2}=2^x[\ln(2)]^2+x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}$. Plugging in $x=0$ and $\frac{dy}{dx}=\frac{2^0\ln(2)+1}{1-0}=\ln(2)+1$, we get $\frac{d^2y}{dx^2}=2^0[\ln(2)]^2+0\frac{d^2y}{dx^2}+(\ln(2)+1)+(\ln(2)+1)\rightarrow\frac{d^2y}{dx^2}=[\ln(2)]^2+2\ln(2)+2$. **D.**

20. The formula for each step of Newton's method is just a rearrangement of the equation of a tangent line to solve for its x intercept. The other answers describe Euler's Method (A), Riemann Sums/integration (C), and polar integration (D). **B.**

21. $\sum_{j=1}^{2021}[(-1)^j(log(j^2x))]=\sum_{j=1}^{2021}[(-1)^j log(x)+(-1)^j log(j^2)]=\sum_{j=1}^{2021}[(-1)^j log(x)]+\sum_{j=1}^{2021}[(-1)^j log(j^2)]$. Since the second sum is a constant, our desired expression becomes $\frac{d}{dx}\left[\sum_{j=1}^{2021}[(-1)^j log(x)]\right]=\frac{d}{dx}[-\log(x)+\log(x)\ldots+\log(x)-\log(x)]=-\frac{1}{x}+\frac{1}{x}\ldots+\frac{1}{x}-\frac{1}{x}=-\frac{1}{x}$. **E.**

22. Assuming the limit exists, $\lim_{x\to0}\frac{6^x-3^x-2^x+1}{x^2}=\lim_{x\to0}\frac{3^x-1}{x}\lim_{x\to0}\frac{2^x-1}{x}=\ln(3)\ln(2)$. Applying L'Hopital twice is also easy. **D.**

23. Let d represent the distance from the phone to the plane of the board (it depends on where Jeffrey stands). Since we are looking for the worst case for each d, we want the height of the board at its vertical midline to exactly fill the angle (otherwise the angle could be made smaller at that d). We can draw a triangle with vertices at the phone (P), the top of the whiteboard in the vertical midline of the picture (T), and the bottom of the whiteboard in the vertical midline of the picture (B). Then, we can draw the altitude from P to $\overline{TB}$ and extend $\overline{TB}$ to meet it at point X. $PX=d, TB=6, BX=8-6=2$, and $\angle TPB$ is the camera's captured angle. In terms of d, the measure of $\angle TPB$ is $\tan^{-1}(\frac{8}{d})-\tan^{-1}(\frac{2}{d})$. By taking its derivative, we find that this quantity has a maximum value of $\tan^{-1}(2)-\tan^{-1}(\frac{1}{2})$ at $d=4$. Since this is the maximum angle that may be necessary, it is the minimum angle the camera must accommodate. **A.**

24. Let the slope of the line be $m<0$. Then the line has equation $y=mx+3-2m$, so it has intercepts of $(0,3-2m)$ and $\left(2-\frac{3}{m},0\right)$. By shoelace, the area of the

triangle is $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-\frac{3}{m} & 4 \\ 3-2m & 0 & 4 \end{vmatrix} = 4-2m-\frac{3}{2m} > 0$. This expression has derivative (with respect to m) $-2+\frac{3}{2m^2}$ and positive second derivative, so the relative minimum occurs at $m=-\frac{\sqrt{3}}{2}$. Plugging this in gives the area as $4+2\sqrt{3}$. **A.**

25. Let $y=f(x)$. Then $\ln(y)=3\ln(x)+x+5\ln(\ln(x))$. Implicitly differentiating gives $\frac{y'}{y}=\frac{3}{x}+1+\frac{5}{x\ln(x)}$. Then $y'(e)=y(e)\left(\frac{3}{e}+1+\frac{5}{e}\right)=e^{e+3}\left(\frac{8}{e}+1\right)=e^{e+3}+8e^{e+2}$. **C.**

26. Since $y=e^x$ is the reflection of $y=\ln(x)$ about the line $y=x$, it follows that the reflection of any line tangent to both graphs about $y=x$ is still tangent to $y=e^x$ and $y=\ln(x)$. This means the product of m_1 and m_2 is 1. **C.**

27. Let $C=(0,c)$, $B=(0,c+1)$, and $\theta=\angle BAC$. Line CA has slope $1-c$ and line BA has slope $-c$. This means $\tan(\theta)=\tan(\arctan(1-c)-\arctan\ (-c))=\frac{1}{c^2-c+1}$. Differentiating with respect to time, we have $\sec^2(\theta)\frac{d\theta}{dt}=-\frac{2c-1}{(c^2-c+1)^2}\frac{dc}{dt}$. When $c=1$, $\theta=\frac{\pi}{4}$, so $\frac{d\theta}{dt}=-\frac{1}{1}\cdot\frac{2}{2}=-1$. Since we just want the speed, the answer is 1. **B.**

28. I is easily true. II and III are also true by rewriting $P(x)$ as $(x-a)(x-b)(x-c)$. II follows by the Product Rule and II and III are equivalent after distributing. **D.**

29. The idea is to look at III from problem 28. Assume $x\neq a,b,c$. Observe that $P'(x)=P(x)\left(\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}\right)$, so $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=\frac{P'(x)}{P(x)}$. Differentiating gives $\frac{-1}{(x-a)^2}+\frac{-1}{(x-b)^2}+\frac{-1}{(x-c)^2}=\left(\frac{P'(x)}{P(x)}\right)'=\frac{P(x)P''(x)-\left(P'(x)\right)^2}{\left(P(x)\right)^2}$. Plugging in $x=1$ gives $\frac{7\cdot 4-3^2}{7^2}=\frac{19}{49}$. The desired expression is then $-\frac{19}{49}$. **A.**

30. Let $f(x)=\frac{x}{1+x^2+x^4}$. Then $f'(x)=\frac{1}{(1+x^2+x^4)^2}\left(1+x^2+x^4-x(2x+4x^3)\right)=-\frac{1}{(1+x^2+x^4)^2}(3x^4+x^2-1)$. Then f has critical points when $3x^4+x^2-1=0$, or when $x^2=\frac{-1+\sqrt{13}}{6}$ (the other root is negative). When $x=\sqrt{\frac{-1+\sqrt{13}}{6}}$, f' changes from positive to negative, so $x=\sqrt{\frac{-1+\sqrt{13}}{6}}$ gives the global maximum (as f tends to zero as x approaches both infinities). Thus $6x^2+1=\sqrt{13}$. **B.**