January Invitational Algebra II Team Solutions

Answer Key:

Question	A	B	C	D
1	-5	$-3/5$	4041	6
2	$-3/4$	7	4	$-2/3$
3	30	7	40	$819/100$ $= 8.19$
4	$3/4$	6	21	0
5	3	128	7	$2/3$
6	7	1	333	15
7	$1/9$	4095	6	729
8	-5	-8	6	$13/2$
9	-12	9	0	-16
10	$400/7$	98	7	π
11	$-16i$	2	$1/58$	$-i$
12	12	44	50	891
13	4097	1267	1679	77
14	315	200	2	30

Question #1

A:

The value of x can quickly be found by using Cramer's Rule.

$$\mathbf{A} = x = \frac{D_x}{D} = \frac{\begin{vmatrix} 0 & -3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & -1 \end{vmatrix}} = \frac{-25}{5} = -5$$

B:

Similar to above, $\mathbf{B} = y = \frac{D_y}{D} = \frac{\begin{vmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 4 \\ 0 & 1 & 2 \\ -1 & 2 & -1 \end{vmatrix}} = \frac{-3}{5}$

C:

$$\mathbf{C} = \begin{vmatrix} 2021 & 2020 \\ 2020 & 2021 \end{vmatrix} = 2021^2 - 2020^2 = (2021 + 2020)(2021 - 2020) = 4041$$

D:

$$\begin{vmatrix} \boldsymbol{D} & 2 & 3 \\ 1 & 2 & 0 \\ 4 & -1 & \boldsymbol{D} \end{vmatrix} = 2\boldsymbol{D}^2 - 2\boldsymbol{D} - 27 = 33 \rightarrow 2\boldsymbol{D}^2 - 2\boldsymbol{D} - 60 = 0 \rightarrow (\boldsymbol{D} - 6)(\boldsymbol{D} + 5) = 0$$

$\mathbf{D} > 0$, so $\mathbf{D} = 6$

Question #2

A:

$6x + 8y = 10 \rightarrow y = -\frac{3}{4}x + \frac{5}{4} \rightarrow \mathbf{A} = -\frac{3}{4}$

B:

Let the slope between the points be m. Then $m = \frac{8-7}{-6-1} = -\frac{1}{7}$, so the slope of the perpendicular line is $-\frac{1}{m} = 7 = \mathbf{B}$

C:

The slope of any line parallel to line k has the same slope as k. $4x - y = -3 \rightarrow y = 4x + 3$, so **C** = 4

D:

The inverse of a line is its reflection across the line $y = x$, so the slopes of the line and its inverse are reciprocals. **D** = $(-\frac{3}{2})^{-1} = -\frac{2}{3}$
Alternatively, solve for x to obtain $f^{-1}(x) = -\frac{2}{3}x + \frac{7}{6}$, so the slope is $-\frac{2}{3}$

Question #3

A:

The existing sample contains $(0.25)(60) = 15$ mL of vinegar. **A** is the amount of 10% vinegar added, so the new total mL of sample will be $60 + \boldsymbol{A}$, the new amount of vinegar will be $15 + \frac{\boldsymbol{A}}{10}$, and the new concentration is given to be 20%. Then $0.2 = \frac{15+\frac{\boldsymbol{A}}{10}}{60+\boldsymbol{A}} \rightarrow 12 + \frac{\boldsymbol{A}}{5} = 15 + \frac{\boldsymbol{A}}{10} \rightarrow \frac{\boldsymbol{A}}{10} = 3 \rightarrow$ **A** $= 30$.

B:

The ratio of mustard to total volume is $\frac{1}{3}$, so **B** $= \frac{1}{3}(21) = 7$

C:

If **C** is the original vinegar concentration of the sample, then the initial amount of vinegar is $20\boldsymbol{C}$. After removing 1 liter of sample and replacing it with pure vinegar, the total volume stays at 20, but the total amount of vinegar is $20\boldsymbol{C} - \boldsymbol{C}(1) + 1 = 1 + 19\boldsymbol{C}$, so the new concentration is $\frac{1+19\boldsymbol{C}}{20}$. This is given to be equal to $43\% = \frac{43}{100}$, so we have $\frac{43}{100} = \frac{1+19\boldsymbol{C}}{20} \rightarrow \frac{43}{5} = 1 + 19\boldsymbol{C} \rightarrow 19\boldsymbol{C} = \frac{38}{5} \rightarrow \boldsymbol{C} = \frac{2}{5}$

D:

The mL of vinegar in the first sample is $(0.6)(9) = 5.4$, and the mL of vinegar in the second sample is $(0.15)(16) = 2.4$, so the total amount of vinegar in the mixture is $5.4 + 2.4 = 7.8$ mL. The mass of the vinegar in grams is the vinegar's volume times its density, so **D** $= (7.8)(1.05) = 8.19$

Question #4

A:

Factoring $f(x)$, we get $f(x) = \frac{(x+3)^2(x-6)}{(2x+9)(x-6)}$. The slant asymptote can be found by dividing $(2x + 9)$ into $(x + 3)^2$, resulting in the line $y = \frac{1}{2}x + \frac{3}{4}$, so the y-intercept $= \frac{3}{4} =$ **A**

B:

A removable discontinuity occurs when a factor is common to both the numerator and denominator of a rational function. In this case, $(x - 6)$ is in both the numerator and the denominator of $f(x)$, so the removable discontinuity occurs at $x = 6 =$ **B**

C:

In a rational function, a horizontal asymptote occurs at $y = 0$ when the degree of the denominator is higher than that of the numerator. The shortest distance from $y = 0$ to $(20, 21)$ is $21 =$ **C**

D:

Factoring, we get $y = \frac{(k+x)(k-x)}{(x+2)(x+1)(x-3)}$, so for $x = 3$ to not be an asymptote, $(x - 3)$ must also be a factor of the numerator. This only occurs when $k = 3$ or -3, so **D** $= 3 + (-3) = 0$

Question #5

A:

$$\sqrt{13 + 4\sqrt{10}} = \sqrt{8 + 5 + 2\sqrt{40}} = \sqrt{(\sqrt{8} + \sqrt{5})^2} = \sqrt{8} + \sqrt{5}$$

A $= |8 - 5| = 3$

B:

$$f(x) = \sqrt{x\sqrt{x\sqrt{x}}} = (x \cdot \left(x \cdot x^{1/2}\right)^{1/2})^{1/2} = x^{\frac{7}{8}}$$

B $= \boldsymbol{f}(256) = (256)^{\frac{7}{8}} = 128$

C:

$(\sqrt[3]{2}+\sqrt[6]{5})(\sqrt[3]{4}-\sqrt[6]{20}+\sqrt[3]{5})$ is in the form $(a+b)(a^2-ab+b^2)$, which is the factored form of a^3+b^3. Thus, $a=\sqrt[3]{2}$ and $b=\sqrt[6]{5}$, so the expression equals $2+\sqrt{5}$. This yields $p=2$ and $q=5$, so **C** $=2+5=7$

D:

D $=\sqrt{\frac{40^4\cdot 5}{20^5\cdot 9}}=\sqrt{(\frac{40}{20})^4\frac{5}{20\cdot 9}}=\sqrt{\frac{16\cdot 5}{20\cdot 9}}=\sqrt{\frac{4}{9}}=\frac{2}{3}$

Question #6

A:

$x^2<4\pi$ → $-\sqrt{4\pi}<x<\sqrt{4\pi}$, and $9<4\pi<16$, so $-\sqrt{9}\le x\le\sqrt{9}$ → $-3\le x\le 3$. This yields **A** $=7$ solutions

B:

We have $x>\sqrt[3]{6}$ and $-\sqrt{5}<x<\sqrt{5}$. $1<6<8$ and $4<5<9$, so for integer x, $x\ge 2$ and $-2\le x\le 2$. This yields only one solution, $x=2$, so **B** $=1$

C:

Let $k=x-9$. Then $-18\le k\le 18$, so the 37 integer solutions for k sum to 0. Substituting $x=k+9$, the 37 solutions for x sum to $9(37)=333=$ **C**

D:

For integer x, $-6\le x\le 8$. The solutions for x where $-6\le x\le 6$ all sum to 0, so the sum of integer solutions for x equals $7+8=15=$ **D**

Question #7

A:

$6x+28\sqrt{x}-10=0$ → $3x+14\sqrt{x}-5=0$ → $(3\sqrt{x}-1)(\sqrt{x}+5)=0$ → $\sqrt{x}=\frac{1}{3},-5$
-5 is an extraneous solution, so $\sqrt{x}=\frac{1}{3}$ → $x=\frac{1}{9}=$ **A**

B:

$2048 = 2^{11}$, so its positive integer factors are $1, 2, 2^2, \dots, 2^{10}$, and 2^{11}
B $= 1 + 2 + 2^2 + \cdots + 2^{11} = \left(\frac{2^{12}-1}{2-1}\right) = 4096 - 1 = 4095$

C:

C $= (\log_3 25)\,(\log_5 27) = (\log_3 5^2)\,(\log_5 3^3) = (2)(\log_3 5)\,(3)(\log_5 3) = 6\left(\frac{\log 5}{\log 3}\right)\left(\frac{\log 3}{\log 5}\right) = 6$

D:

D $= \sqrt[3]{\frac{3^{\left(3^3\right)}}{\left(3^3\right)^3}} = \sqrt[3]{\frac{3^{27}}{3^9}} = \sqrt[3]{3^{18}} = 3^6 = 729$

Question #8

A:

$4x^2 - y^2 + 24x - 4y + 16 = 0 \rightarrow 4(x+3)^2 - (y+2)^2 = -16 + 36 - 4 = 16 \rightarrow$
$\frac{(x+3)^2}{4} - \frac{(y+2)^2}{16} = 1$
Due to the symmetry of the hyperbola, the midpoint between its foci is the center, $(-3, -2)$
A $= -3 + (-2) = -5$

B:

Using the equation from part **A**, the hyperbola is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a = 2$ and $b = 4$. The asymptotes of a hyperbola in this form are $y - k = \pm\frac{b}{a}(x - h)$, so the asymptote with negative slope is $y + 2 = -2(x + 3) \rightarrow y = -2x - 8$, which has y-intercept $-8 =$ **B**

C:

$y^2 + 6x - 6y - 21 = 0 \rightarrow (y-3)^2 = -6x + 21 + 9 \rightarrow (y-3)^2 = -6(x-5)$
This is a parabola in the form $(y-k)^2 = -4p(x-h)$, so the length of its latus rectum is $4p = 6 =$ **C**

D:

For a parabola in the form $(y-k)^2 = -4p(x-h)$, the distance from the vertex to the directrix is p. Using the equation from part **C**, $-4p = -6$ → $p = 3/2$. The directrix is a vertical line located to the right of the vertex, so it has equation $x = 5 + \frac{3}{2} = \frac{13}{2} =$ **D**

Question #9

All solutions use a polynomial in the form $p(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$

A:

Use Vieta's formula for the sum of roots taken two at a time:
A $= \frac{c}{a} = \frac{-36}{3} = -12$

B:

$$\frac{1}{rs} + \frac{1}{st} + \frac{1}{rt} = \frac{r+s+t}{rst}$$

By Vieta's formulas, **B** $= \frac{-\frac{b}{a}}{-\frac{d}{a}} = \frac{-\frac{9}{3}}{-\frac{1}{3}} = 9$

C:

Use Vieta's formula for the sum of roots:
C $= -\frac{b}{a} = -\frac{0}{1} = 0$

D:

$h(x) = x^5 - 3x^3 - 9x^2 + 4x + 16$
Use Vieta's formula for the product of roots:
D $= -\frac{f}{a} = -\frac{16}{1} = -16$

Question #10

A:

After 10 seconds, Cardi B has made it to the pole and begins turning back, and Patty G is still 60 yards away. Since Patty G runs at 4 yd/s and Cardi B runs at 10 yd/s, the ratio of the distances they now each run before they meet is $4 : 10$. Excluding the first 10 seconds, let Patty G's distance be $4x$ and let Cardi B's distance be $10x$. Then $4x + 10x = 60 \rightarrow x = 30/7$, so Patty G's total distance is $40 + 4x = 40 + 120/7 = 400/7$ = **A**

B:

Let P be the hiker's usual rate, and let R be the effect of an additional ramen brick on his rate. Since rate = distance / time, we have the equations $P + 2R = 980/70 = 14$ and $P - R = 980/49 = 20$, which yield $P = 18$ and $R = -2$. Then **B** $= \frac{980}{P+4R} = \frac{980}{10} = 98$

C:

Work = rate · time, so Liszt plays (15 notes/second) · (60 seconds) = 900 notes total. Then Chopin plays $900 - 480 = 420$ notes total, so Chopin's rate in notes per second is $\frac{420}{60} = 7 =$ **C**

D:

$x^2 + y^2 + 18x - 2y + 66 = 0 \rightarrow (x+9)^2 + (y-1)^2 = -66 + 81 + 1 = 16$. This is a circle with radius 4, so his distance is $2\pi(4) = 8\pi$. Time = distance / rate, so **D** $= 8\pi/8 = \pi$

Question #11

A:

A $= (1+i)^3(1-i)^5 = ((1+i)^3(1-i)^3)(1-i)^2 = \left((1+i)(1-i)\right)^3(1-i)^2 = (1+1)^3(1-i)^2 = (8)(-2i) = -16i$

B:

$$\frac{25}{2i-1} + \frac{25}{(2i-1)^2} = \frac{25(2i-1)+25}{(2i-1)^2} = \frac{50i}{(2i-1)^2} = \frac{50i}{-3-4i} = \frac{50i(-3+4i)}{(-3-4i)(-3+4i)}$$
$$= \frac{50i(-3+4i)}{9+16} = 2i(-3+4i) = -8-6i$$

B $= b - a = -6 - (-8) = 2$

C:

$$\sqrt{\boldsymbol{C}} = \frac{z \cdot \bar{z}}{|z|^3} = \frac{(3-7i)(3+7i)}{(\sqrt{3^2+7^2})^3} = \frac{3^2+7^2}{(3^2+7^2)\sqrt{3^2+7^2}} = \frac{1}{\sqrt{3^2+7^2}} = \frac{1}{\sqrt{58}} \rightarrow \mathbf{C} = \frac{1}{58}$$

D:

Note that the powers of $-i$ repeat, with a period of 4:

$(-i)^1 = -i$, $(-i)^2 = -1$, $(-i)^3 = i$, $(-i)^4 = 1$, $(-i)^5 = -i$, …
$2021^2 = (2020 + 1)^2 = 2020^2 + 2(2020) + 1$, and 2020 is divisible by 4, so 2021^2 leaves a remainder of 1 when divided by 4. Thus, **D** = $(-i)^{2021^2} = (-i)^1 = -i$

Question #12

A:

Let x equal the number of years after the current year. Then we have $20 \cdot 2^{x/2} = 1280$ → $2^{x/2} = 64$ → $x/2 = 6$ → $x = 12 =$ **A**

B:

$44^2 = 1936$ and $45^2 = 2025$, so 44^2 is the greatest square less than 2021. Thus, there are **B** = 44 positive integer squares under 2021

C:

There are 5 ways to pick 1 base ingredient out of 5, and there are $\frac{5 \cdot 4}{2 \cdot 1} = 10$ ways to pick 2 toppings out of 5. **C** = $5 \cdot 10 = 50$

D:

D = $1000\left(\frac{9}{10}\right)\left(\frac{11}{10}\right)\left(\frac{9}{10}\right) = 1000\left(\frac{891}{1000}\right) = 891$

Question #13

A:

The first card can be any color, and the next 6 must be that same color, with an equal probability of $\frac{1}{4}$ for each draw. Thus, the probability that all 7 draws yield the same color is $1 \cdot \left(\frac{1}{4}\right)^6 = \frac{1}{4096}$
A = $1 + 4096 = 4097$

B:

30 of the 40 cards are not blue, so the probability for each of the 5 cards that it is not blue is $\frac{3}{4}$. Thus, the probability that none of the 5 draws yield blue cards is $(\frac{3}{4})^5 = \frac{243}{1024}$

B $= 243 + 1024 = 1267$

C:

Consider the complement, the probability that the green 5 is not drawn in either draw. There is only one green 5 in the 40-card deck, so the probability it is not drawn in a single draw is $\frac{39}{40}$, and the probability it is not drawn in either of two draws is $(\frac{39}{40})^2$. Thus, the probability that it **is** drawn at least once in two draws is $1 - \left(\frac{39}{40}\right)^2 = \frac{40^2-39^2}{40^2} = \frac{(40+39)(40-39)}{1600} = \frac{79}{1600}$
C $= 79 + 1600 = 1679$

D:

This is equivalent to drawing 2 cards without replacement. There are 4 cards displaying 1 out of the 40 cards initially, and afterwards there are 3 cards displaying 1 out of the 39 remaining cards. Thus, the probability of the first card displaying 1 and the second card not displaying 1 is $\left(\frac{4}{40}\right)\left(\frac{39-3}{39}\right) = \frac{6}{65}$. This probability must be multiplied by 2 to account for the other permutation (where the second card displays 1 and the first does not), so the overall probability is $\frac{12}{65}$
D $= 12 + 65 = 77$

Question #14

A:

Consider the complement, the nonnegative integers under 500 with a sum of digits less than 10. When the hundreds digit is 0, this includes the integers (grouped by tens digit) $0 - 9$, $10 - 18$, ... , $80 - 81$, and 90, which is $10 + 9 + \cdots + 2 + 1 = 55$ integers. When the hundreds digit increases by 1, the last integer in each of the tens-digit groups no longer yields a sum of digits less than 10, so the number of cases decreases by the number of tens-digit groups. Adding up the cases for each hundreds digit under 5, this results in $55 + 45 + 36 + 28 + 21 = 185$ nonnegative integers under 500 that have a sum of digits less than 10; thus, the number of positive integers under 500 with a sum of digits greater than or equal to 10 is $500 - 185 = 315 =$ **A** (0 is included in both the 500 and the 185, so it doesn't matter that we considered nonnegative integers rather than positive)

B:

Clearly, the greatest exponent for a power of 11 that divides 2021! is less than the greatest exponent for a power of 2 that divides 2021!. Thus, we need the greatest value of n for which 11^n divides $2021! = (2021)(2020)(2019) \ldots (2)(1)$. $\lfloor 2021/11 \rfloor = 183$, so there are 183 multiples of 11 under 2021; $\lfloor 183/11 \rfloor = 16$, so there are 16 multiples of 11^2; and $\lfloor 16/11 \rfloor =$

1, so there is 1 multiple of 11^3. Therefore, the greatest value of n for which 11^n divides this product and thus $2021!$ is $183 + 16 + 1 = 200 =$ **B**

C:

If two integers sum to 1853, one must be odd and one must be even. Without loss of generality, let x^2 be even and let y^2 be odd. x and y are prime, so if x^2 is even, x must be 2. Thus, $x^2 = 4$, $y^2 = 1849$, and $y = 43$. This yields **C** $= 2$ solutions: $(2, 43)$ and $(43, 2)$

D:

For an integer I with prime factorization ${a_1}^{b_1}{a_2}^{b_2}{a_3}^{b_3} \ldots {a_n}^{b_n}$, the number of positive integer divisors of I is $(b_1 + 1) \cdot (b_2 + 1) \cdot (b_3 + 1) \cdot \ldots \cdot (b_n + 1)$. This product is equal to 8, so the product could either be $2 \cdot 2 \cdot 2$, $4 \cdot 2$, or 8, necessitating an integer with prime factorization of the form abc, a^3b, or a^7. Minimizing for each of these cases:

Case 1: abc

$2 \cdot 3 \cdot 5 = 30$, $2 \cdot 3 \cdot 7 = 42$, $2 \cdot 3 \cdot 11 = 66, \ldots$

Case 2: a^3b

$2^3 \cdot 3 = 24$, $2^3 \cdot 5 = 40$, $3^3 \cdot 2 = 54, \ldots$

Case 3: a^7

$2^7 = 128, \ldots$

The second smallest of these integers is $30 =$ **D**