1 Answers

1. (B)
2. (A)
3. (A)
4. (B)
5. (B)
6. (D)
7. (B)
8. (A)
9. (A)
10. (D)
11. (D)
12. (C)
13. (C)
14. (C)
15. (B)
16. (C)
17. (A)
18. (B)
19. (D)
20. (C)
21. (B)
22. (C)
23. (C)
24. (C)
25. (A)
26. (D)
27. (D)
28. (D)
29. (B)
30. (C)

2 Solutions

1. Saathvik must flip three tails, then heads. This happens with probability $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
(B)

2. Corbin first draws one sock. With probability $\frac{4}{5}$, the second sock he draws will not match the first sock. With probability $\frac{2}{4}$, the third sock he draws matches neither the first sock nor the second sock. Therefore, with probability $\frac{4}{5}\left(\frac{2}{4}\right) = \frac{2}{5}$, the three socks will all be different colors.
(A)

3. The probability is $\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} = \frac{2}{3}$
(A)

4. There are a total of 8 kings and jacks, each of which is distinguishable from the others. Thus, the number of hands that King Jake likes is 8 choose 5 = 56
(B)

5. The empirical rules states that 95 percent of the values are within two standard deviations of the mean. Half of those outside that value will be above, so 0.05/2=0.025
(B)

6. The standard deciation of the binomial distribution is $\sqrt{npq}$
(D)

7. There are 21 prime numbers between 1 and 75, so the correct probability is
$$\frac{21}{75} = \frac{7}{25}$$
(B)

8. The probability Daniel wins exactly two out of his three matches is:
$$\binom{3}{2} \cdot \left(\frac{7}{10}\right)^2 \cdot \frac{3}{10} = 0.441$$
(A)

9. Treat the "block" of 2 Es as a single letter and the "block" of 2 Fs as a single letter. Then, there are simply 5! = 120 possible permutations.
(A)

10. The probability that Curry makes all three free throws is $\left(\frac{9}{10}\right)^3$. The probability that Curry makes at least two free throws is $\left(\frac{9}{10}\right)^3 + 3 \cdot \left(\frac{9}{10}\right)^2 \cdot \frac{1}{10} = \frac{972}{1000}$. Dividing, the conditional probability is $\frac{729}{972} = \frac{3}{4}$.
(D)

11. Note that removing the top 26 cards without looking has no effect on the probability, so the problem is equivalent to drawing the first card from the deck. Thus, the answer is simply $\frac{1}{13}$.
(D)

12. All 4 numbers must be between 0.4 and 0.8, so the answer is

$$\left(\frac{2}{5}\right)^4 = \frac{16}{625}$$

(C)

13. Careful casework shows the only valid partitions are $5+2, 4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1$. There are $\boxed{5}$.
(C)

14. Consider the function $dp(x)$ which returns the answer for an arbitrary x. Note that $dp(0) = 1$ and $dp(x) = dp(x-2) + dp(x-3) + dp(x-5)$. Some computation gives $dp(10) = dp(5) + dp(7) + dp(8) = 3 + 5 + 6 = \boxed{14}$.
(C)

15. There are $\binom{10}{4} = 210$ total ways to elect a committee. A committee is valid if both Andy and Buffy are on it or if neither of them are on it. In the former case, there $\binom{8}{2} = 28$ ways to choose the other members. In the latter case, there are $\binom{8}{4} = 70$ ways to choose the other members. Thus, the final answer is

$$\frac{28+70}{210} = \frac{98}{210} = \boxed{\frac{7}{15}}$$

(B)

16. We use Burnside's lemma to account for double-counting. We calculate the number of fixed points for each element of the cyclic group of order 6. For rotating by $0°$, there are 3^6 fixed points as this is the identity transformation. Moreover, note that we can take advantage of symmetry because rotating by $60°$ and $300°$ will have the same number of fixed points: 3^1. Likewise, rotating by $120°$ and $240°$ both have 3^2 fixed points, and rotating by $180°$ gives 3^3 fixed points. Thus, our final answer is

$$\frac{3^6 + 2 \cdot 3^2 + 2 \cdot 3 + 3^3}{6} = \boxed{130}$$

(C)

17. Let D_n be the number of dearrangments of n numbers. We can calculate recursively that $D_n = (n-1)(D_{n-1} + D_{n-2})$ or simply check by hand that $D_4 = 9$. It follows that our answer is

$$\frac{\binom{6}{2} D_4}{6!} = \boxed{\frac{3}{16}}$$

(A)

18. First note that the sum of all numbers from 1 to 6 is odd so there is no assignment which results in the expression evaluating to 0. Moreover, any expression with negative value can be made to have positive value by simply replacing all of the pluses with minuses and all of the minuses with pluses. Thus, the answer is simply the total number of ways (2^6) divided by 2.
(B)

19. Let p be the probability of winning initially, let p_H be the probability of winning after 1 head has been flipped, and let p_T be the probability of winning after 1 tail has been flipped. Then,

$$p = \frac{p_H + p_T}{2},\ p_H = \frac{1}{8} + \frac{7}{8}p_T,\ p_T = \frac{1}{4}0 + \frac{3}{4}p_H$$

Thus, $p_H = \frac{4}{11}, p_T = \frac{3}{11}, p = \frac{7}{22}$ and the answer is $7 + 22 = \boxed{29}$.
(D)

20. Let F_n be the number of ways Sean can pick his numbers given that there are n such e_i. First, note that $F_1 = 2$, $F_2 = 3$. We wish to compute F_{10}. Note that if e_{10} is 0, e_9 must be 0 and there are F_8 ways to pick the remaining numbers. If e_{10} is 1, there are F_9 ways. Thus, $F_{10} = F_9 + F_8$ and the answer is just the 12th Fibonacci number or 144.
(C)

21. Let the number of steaks Jeffrey, Konwoo, Andrew, and Daniel receives be x, y, z, w respectively. We must have $x > y$ and $x+y+z+w = 12$. Note that the number of quadruples (x, y, z, w) which satisfy this is the same as the number of quadruples (x, y, z, w) which satisfy $x < y$ and $x+y+z+w = 12$. Thus, we count the total number of ways $x+y+z+w = 12$, subtract out the cases when $x = y$, and divide by 2. The total number of ways by stars and bars is $\binom{15}{3} = 455$. When, $x = y$, we must have $2x+y+z = 12$. x can be any integer from 0 to 6. The total number of cases is $13+11+9+7+5+3+1 = 49$. Thus, our final answer is
$$\frac{455-49}{2} = \boxed{203}$$
(B)

22. Note that the probability Kirby gets any question which has actual answer T correct is $\frac{7}{20}$. Similarly, the probability he gets any question which has actual answer F correct is $\frac{13}{20}$. Thus, by linearity, the answer is
$$7 \cdot \left(\frac{7}{20}\right) + 13 \cdot \left(\frac{13}{20}\right) = \frac{109}{10}$$
(C)

23. Note that the 13 Fs must fill the 8 gaps created by the 7 Ts. The expected number of Fs which will fill the first gap is just $\frac{13}{8}$, so the answer is $\frac{13}{8} + 1 = \frac{21}{8}$.
(C)

24. We complementary count. The total number of permutations is $\frac{12!}{2!3!2!}$. By the principle of inclusion-exclusion, the total number of permutations with at least one block of consecutive As is
$$\frac{11!}{2!2!} - \frac{10!}{2!2!}$$
Thus, our final answer is
$$\frac{\frac{12!}{24} - \frac{11!}{4} + \frac{10!}{4}}{\frac{12!}{24}} = \boxed{\frac{6}{11}}$$
(C)

25. Subtract 2 from each of the card numbers so that the problem is equivalent to finding the number of ways Stanley can draw 1 card from each of four colors (with cards from 1 to 9) such that the sum of the numbers on his cards is 21. The number of ways to distribute 21 indistinguishable ones to 4 distinguishable piles (such that each pile receives at least 1 one) is $\binom{20}{3}$. However, this counts impossible configurations such as 14 3 1 1. Note that at most one pile can receive more than 9 ones (because $10 + 10 + 1 + 1 > 21$). There are 4 ways to choose this pile and we "translate" this pile by 9 as well. Thus, the total number of impossible cases is $4\binom{11}{3}$. The final answer is $\binom{20}{3} - 4\binom{11}{3} = 480$.
(A)

26. We do casework based on what the median is. If it's 1, there is one possibility. If it's 2, there are three possibilities. More generally, there are $2k - 1$ possibilities if the median is k. Thus, the answer is
$$\sum_{k=1}^{6} \frac{k(2k-1)}{36} = \boxed{\frac{161}{36}}$$

(D)

27. Consider the generating function
$$g(x) = x(x+1)^{2020}$$
Since 2020 is 1 mod 3, it suffices to compute
$$\frac{g(1)+g(\omega)+g(\omega^2)}{3} \cdot \frac{1}{2^{2020}} = \frac{2^{2020}+2}{3 \cdot 2^{2020}} = \frac{2^{2019}+1}{3 \cdot 2^{2019}}$$
Thus, the answer is $2019 + 1 + 3 + 2019 = \boxed{4042}$.
(D)

28. We claim more generally that the expected length is ab for stopping at a units to the left or b units to the right. One way to see this is to note that
$$E(a,b) = \frac{E(a-1,b+1)+E(a+1,b-1)}{2} + 1$$
An alternative solution can be derived using the fact that the expected amount of time is invariant regardless of the current position. Either way, the answer can be found to be $\boxed{30}$.
(D)

29. We consider cases based on the number of moves Billy makes in each direction. There are three cases to consider. In the first, Billy makes all 6 of his moves in one direction. These 6 moves must be some permutation of $U_xU_xU_xD_xD_xD_x$ (where U_x is a move in a positive direction, D_x is a move in a negative direction). Since there are 4 ways to pick the direction, the first case contributes $4 \cdot \binom{6}{3} = 80$ ways.

In the second case, Billy makes 2 moves in each of 3 directions and no moves in the fourth. There are 4 ways to pick the direction he makes no moves in. Then, his six moves are in the form $U_xD_xU_yD_yU_zD_z$. This case contributes $4 \cdot 6! = 2880$ ways.

The third and final case is when Billy makes 4 moves in one direction, 2 moves in another, and 0 moves in the other two directions. His six moves are in the form $D_xU_xD_yU_yD_yU_y$. This case contributes $\frac{4!}{2} \cdot \binom{6}{2,2,1,1} = 2160$.

In total, there are 5120 ways, so the probability is $\frac{5120}{8^6}$. This is equivalent to the final answer of $\frac{5}{256}$.
(B)

30. Consider the modified question where Bob has an extra move. Then, note that after every person moves, the vector determined by the difference in Billy and Bob's positions changes by 1 unit in one of the four directions randomly. Thus, the modified question is just equivalent to a random walk in two dimensions returning to the origin after 8 moves. Clearly, there must the same number of down and up moves as well as left and right moves. Now, instead of considering each move as left, up, down, or right, we check whether it is LU (left or up) or RU (right or up). If it is LU and not RU, then the move must be left. If it is LU and RU, then the must be up. If it is not LU and RU, the move must be right. If it is not LU or RU, then the move must be down. So, we can see these classifications are equivalent. There are $\binom{8}{4}^2$ ways to assign the LU and RU labels. Thus, the answer to our modified problem is $\frac{\binom{8}{4}^2}{4^8}$. However, in our original problem, we want them to be 1 unit apart, so we must multiply by 4. Our final answer is
$$\frac{\binom{8}{4}^2}{4^7} = \boxed{\frac{1225}{4096}}$$

(C)