NOTA denotes "None of the Above." $\lfloor x \rfloor$ **means the greatest integer** $\leq x$**.** $\lceil x \rceil$ **means the least integer** $\geq x$**. For example,** $\lfloor \pi \rfloor = \lceil e \rceil = 3$**.**

1. Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$$

Find $F(2)$.

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{2}$ (E) NOTA

2. Given that $a_1 = a_2 = 1$ and $a_n = a_{n-1} + 2a_{n-2}$, what is $a_{2021} + a_{2020}$?

(A) 2^{2023} (B) 2^{2022} (C) 2^{2021} (D) 2^{2020} (E) NOTA

3. Let $S = \sum_{n=1}^{\infty} \arctan\left(\frac{F_{n+1}}{F_n F_{n+2} + 1}\right) \arctan\left(\frac{1}{F_{n+1}}\right)$, where F_n is the n^{th} Fibonacci number, defined as $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. S can be written in the form $\frac{\pi^a}{b}$, where $a, b \in \mathbb{N}$. Find $a + b$. Hint: $\tan(x - y) = \frac{\tan(x) - \tan(y)}{\tan(x)\tan(y) + 1}$

(A) 8 (B) 18 (C) 20 (D) 38 (E) NOTA

4. An icosahedron is a regular polyhedron with 12 vertices, 20 faces, and 30 edges. How many distinct rigid rotations are there for an icosahedron in three-dimensional space?

(A) 48 (B) 55 (C) 60 (D) 65 (E) NOTA

5. The Towers of Hanoi is a famous problem. Do you know it? Given a stack of 15 disks on a pole, arranged from largest to smallest on top, together with two empty poles, what is the minimum number of moves required to move the stack of disks from one pole to another? A "move" means moving the top disk from one pole to any other pole, provided a larger disk is never placed on top of a smaller one. Hint: Construct a recursion.

(A) 105 (B) 32767 (C) 14348906 (D) 309435327 (E) NOTA

6. Let $f(x) = x^3 + 18x^2 + 108x + 210$. Let r be the only real root of $f(f(f(f(f(f(x)))))) = 2020$. r can be written in the form $-a + \sqrt[n]{k}$ where $a, n, k \in \mathbb{N}$ and $n > 1$. Find $a + n + k$.

(A) 2761 (B) 2762 (C) 2763 (D) 2764 (E) NOTA

7. Five students at a the Mu Alpha Theta Convention mixer remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2}$ (E) NOTA

8. Evaluate $\lfloor |\cos(4769°)| + |\sin(4769°)| \rfloor$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

9. Find the number of possible finite sequences of real numbers $\{x_n\}_{n=1}^{2020}$ such that

$$\sum_{n=1}^{2020} x_n^2 = \sum_{n=1}^{2020} x_n^3 = \sum_{n=1}^{2020} x_n^4$$

Hint: $Y - 2Y + Y = 0$ for any Y.

(A) 1 (B) 2^{2020} (C) 3^{2020} (D) 2^{4040} (E) NOTA

10. In preparation for the annual FAMAT State Convention, Iris is undergoing a new training regime. However, she has procrastinated on training for too long, and now she only has exactly three weeks to train. Iris has decided to train for 45 hours. She spends a third of the time training during the second week as she did during the first week, and she spends a half of the time training during the third week as during the second week. How much time did she spend training during the second week?

(A) 5 (B) 8.5 (C) 10 (D) 13.5 (E) NOTA

11. The Tribonacci numbers T_n are defined as follows: $T_0 = 0, T_1 = 1$, and $T_2 = 1$. For all $n \geq 3$, we have $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. Compute the smallest Tribonacci number that is both greater than 100 and is prime.

(A) 927 (B) 1705 (C) 13 (D) 149 (E) NOTA

12. How many positive integers $n \leq 2020$ can be written in the form $\sum_{k=0}^{\infty} a_k 3^k$ where $a_k \in \{0, 1\}$?

(A) 127 (B) 128 (C) 255 (D) 256 (E) NOTA

13. Evaluate $\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^{3n}$.

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$ (E) NOTA

14. Let $x_n = 2019 \cdot 5^n - 2020 \cdot 4^n$ for all positive integers n. There exists two real numbers a, b, such that $x_{n+2} = ax_{n+1} + bx_n$ for all positive integers n. Find $a - b$.

(A) 29 (B) 34 (C) 51 (D) 102 (E) NOTA

15. Let $S = \{1, 2, 3, \ldots, 2020\}$. Say a permutation π of S has a *charm* at $k \in S$ if

    (i) $\pi(k) > \pi(k+1)$ for $k = 1$

    (ii) $\pi(k) > \pi(k+1)$ and $\pi(k-1) < \pi(k)$for $1 < k < 2020$

    (iii) $\pi(k-1) < \pi(k)$ for $k = 2020$

    where $\pi(k)$ is the value at the k^{th} spot in the permutation. Let A be the average number of charms over all permutations of S. Find $\lceil A \rceil$.

    (A) 337 (B) 505 (C) 674 (D) 1010 (E) NOTA

16. Evaluate $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{2^n3^m}$.

    (A) $\frac{11}{6}$ (B) 2 (C) $\frac{13}{6}$ (D) 3 (E) NOTA

17. Find the number of positive integers $n \leq 2020$ such that $P(x) = x^{n+1} + x^n + 1$ is divisible by $x^2 + x + 1$.

    (A) 337 (B) 505 (C) 674 (D) 1010 (E) NOTA

18. In how many ways can you list the integers $1, 2, 3, 4, 5, 6, 7, 8$ in a row such that each integer is either greater than all its predecessors or less than all its predecessors?

    (A) 96 (B) 112 (C) 132 (D) 156 (E) NOTA

19. Saathvik and Sharvaa are math competition students, so they enjoy researching math problems.They each are able to do math problems at a constant rate. One day, Dr. Santos gives them a problem set to do. Working alone, Saathvik can solve all the problems in 6 hours, while Sharvaa can solve them in 8 hours. When they work together, they are more efficient because they are able to discuss the problems, so their combined output is the sum of their individual outputs plus 2 additional problems per hour. Working together, they complete the problem set in 3 hours. How many problems are on the problem set?

    (A) 36 (B) 48 (C) 52 (D) 60 (E) NOTA

20. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 2021 are both prime and boring?

    (A) 5 (B) 6 (C) 7 (D) 8 (E) NOTA

21. Find the smallest prime that divides $n^2 + 3n + 13$ for some integer n.

    (A) 3 (B) 5 (C) 7 (D) 11 (E) NOTA

22. How many solutions (a, b, c) in positive integers are there such that $4a^3 + 2019b^3 = c^4$?

    (A) 0 (B) 1 (C) 2015 (D) 2019 (E) NOTA

23. Rohan is running, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

(A) 12 (B) 13 (C) 14 (D) 15 (E) NOTA

24. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10 percent. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

(A) $\frac{6}{5}$ (B) $\frac{11}{10}$ (C) 1 (D) $\frac{10}{11}$ (E) NOTA

25. How many digits does the number 999^{999} have?

(A) 2991 (B) 2995 (C) 2997 (D) 3000 (E) NOTA

26. Let $P = \prod_{n=0}^{\infty} \left(\frac{1}{2020^{2^n}} + 1 \right)$. P can be written as a fraction $\frac{a}{b}$ where $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Find $a + b$. Hint: Consider $(1 + x)(1 + x^2)(1 + x^4) \dots$ as a sum

(A) 4039 (B) 4040 (C) 4041 (D) 4042 (E) NOTA

27. Compute the smallest positive integer $x > 100$ such that every permutation of the digits of x is prime.

(A) 111 (B) 151 (C) 149 (D) 113 (E) NOTA

28. Find the smallest positive integer that leaves a remainder of 1 when divided by 2, a remainder of 3 when divided by 4, a remainder of 5 divided by 6, a remainder of 7 divided by 8, and a remainder of 9 divided by 10.

(A) 119 (B) 239 (C) 299 (D) 359 (E) NOTA

29. We define a positive integer p to be almost prime if it has exactly one divisor other than 1 and p. Compute the sum of the three smallest numbers which are almost prime.

(A) 14 (B) 70 (C) 38 (D) 29 (E) NOTA

30. Yesterday was my birthday. I just realized that my age is the sum of the 4 digits of the year I was born in. How old am I? Assume I was born in the 21st century.

(A) 5 (B) 11 (C) 27 (D) 64 (E) NOTA