Answers: `CBCCA DCBAD CDBBD BDBAA EDABB AEDAB`

1) It takes 12 terms for the series to increase by 48, so the common difference is 4. Solving $4 \times 9 + n = 13$ gives the 1^{st} term as equaling -23, so the n^{th} term of the sequence is $4n - 23$ and the 2021^{st} term is 8061.

2) By partial fractions, $\dfrac{1}{k^2+3k} = \dfrac{1}{3}\left(\dfrac{1}{k} - \dfrac{1}{k+3}\right)$. In the sum, all terms except the first three will cancel out, so the answer is $\dfrac{1}{3}\left(\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{3}\right) = \dfrac{11}{18}$.

3) The three terms are asymptotically equivalent to $\sqrt{n}$, $3\sqrt{n}$, and $4\sqrt{n}$ respectively, and the constants added will not affect the value of the limit, which is equal to $(1+3-4)\sqrt{n} = 0$.

4) Since all terms in the sequence are rational powers of 2, consider $\{b_n\}_{b\geq 1}$ such that $b_n = \log_2 a_n\ \forall n$. Each term in b_n is the arithmetic mean of the previous two terms, so the difference between the first two terms $(+1)$ will be multiplied by $-\dfrac{1}{2}$ for every successive pair of terms. The limit of b_n is $1 + \displaystyle\sum_{i=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1 + \frac{2}{3} = \frac{5}{3}$, so $\displaystyle\lim_{n\to\infty} = 2^{\frac{5}{3}}$.

5) Note that if a_i is negative, then a_{i+1} will be more large negative than a_i and the sequence will diverge to $-\infty$. Solving $2a_n - a_n^2 < 0$ also gives $a_n > 2$ as values that produce negative numbers in successive terms. This leaves $k \in [0,2]$ as the valid initial values.

6) The second term of the sum is a geometric series with first term $-\dfrac{2}{27}$ and common ratio $\dfrac{1}{3}$, so its sum is $-\dfrac{2/27}{2/3} = -\dfrac{1}{9}$.

   $\displaystyle\sum_{k=1}^{\infty}\frac{k}{3^k}$ is the sum of an infinite number of geometric series with common ratio $\dfrac{1}{3}$ and first term one-third of the previous, starting with $\dfrac{1}{3}$. The first series has sum $\dfrac{1/3}{2/3} = \dfrac{1}{2}$ so the sum of all of the series is $\dfrac{1/2}{2/3} = \dfrac{3}{4}$. The given sum is missing the $k=1$ term, so the first half of the given series sums to $\dfrac{3}{4} - \dfrac{1}{3} = \dfrac{5}{12}$. The overall sum is $\dfrac{5}{12} - \dfrac{1}{9} = \dfrac{11}{36}$.

7) The sequence is always positive and has a limit at 0, so it is bounded. The derivative of the function is $-\dfrac{n^2(n^4-6063)}{(n^4+2021)^2}$, which is positive until $n^4 > 6063$, or $n \geq 9$. Thus, it is not monotonic overall, despite being monotonically decreasing for $n \geq 9$.

8) Let $a_n = \left(1+\dfrac{1}{n}\right)^n$. Then $e - a_n \geq a_{nk} - a_n = \left(1+\dfrac{1}{nk}\right)^{nk} - \left(1+\dfrac{1}{n}\right)^n \geq \dfrac{k-1}{2nk}\left(1+\dfrac{1}{n}\right)^{n-1}$. Limiting at infinity, we obtain $\displaystyle\lim_{n\to\infty} n(e-a_n) \geq \frac{e}{2}$. The same can be demonstrated with

the sequence $b_n = \left(1+\frac{1}{n}\right)^{n+1}$ to prove $\lim_{n\to\infty} n(b_n - e) \geq \frac{e}{2}$.

$\lim_{n\to\infty} n(b_n - a_n) = \lim_{n\to\infty} n\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}-1\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, so we have $\lim_{n\to\infty} n(e-a_n) = \lim_{n\to\infty}(b_n - e) = \frac{e}{2}$.

9) Explicitly, $a_n = \frac{1-3^{-n}}{2}$, so $\frac{1}{2} - a_n = \frac{1}{2\cdot 3^n}$. The sum is of a geometric series with first term $\frac{1}{6}$ and common ratio $\frac{1}{3}$, so its value is $\frac{1/6}{2/3} = \frac{1}{4}$.

10) Using the Stirling approximation $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, the central binomial coefficient is asymptotic to $\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2\pi n\left(\frac{n}{e}\right)^{2n}} = \frac{2^{2n}}{\sqrt{\pi n}} = \frac{4^n}{\sqrt{\pi n}}$.

11) $\sum_{p=1}^{2000}\sum_{q=1}^{2000}(p+q) = \sum_{p=1}^{2000}\left(2000p + \frac{2000\cdot 2001}{2}\right) = 2000\cdot\frac{2000\cdot 2001}{2} + 2000\cdot\frac{2000\cdot 2001}{2} =$ $2000\cdot 2000\cdot 2001 = 8,004,000,000$. The sum of the digits of this is 12.

12) Note that the Maclaurin series for $1-\cos x$ starts $\frac{x^2}{2} - \frac{x^4}{24} + \ldots$, so we only need to examine the x^2 of the Maclaurin series of the numerator. e^x and $\sec^2 x$ both have a leading term of 1, so the x^2 term of the numerator has coefficient 1. The limit is $\frac{1}{1/2} = 2$.

13) $f'(x) = -\frac{1}{x^2}$, so $f'(1) = -1$ and $p(x) = -x+2$. $f''(x) = \frac{2}{x^3}$ and $f'''(x) = -\frac{6}{x^3}$, so $f''(1) = 2$ and $f'''(1) = -6$. Thus, the third-degree Taylor series for $\frac{1}{x}$ is $q(x) = 1-(x-1)+(x-1)^2-(x-1)^3$. $A = p(0) = 2$ and $B = q(0) = 4$, so $A - B = -2$.

14) Since $\frac{1}{n}$ is decreasingly convergent to 0, Dirichlet's test can be used to prove convergence if $S_n = \sum_{k=1}^{n}$ is bounded. $\mathrm{d}S_n = \Im\left(\sum_{k=1}^{n} e^{ik}\right)$. Since $|\Im(z)| \leq |z|$, $|S_n| \leq e^i \cdot \frac{1-e^{in}}{1-e^i} \leq \frac{2}{|1-e^i|} < \infty$. Thus, Dirichlet's test applies.

When checking for absolute convergence, note that $|\sin n| + |\sin(n+1)| > \epsilon$, so $\sum_{n=1}^{\infty}\frac{|\sin n|}{n} \geq \sum_{n=1}^{\infty}\frac{\epsilon}{2n}$, so the series is only conditionally convergent.

15) For A) and B), the limit-comparison test can be used, since the harmonic series is divergent. $\lim_{n\to\infty}\frac{\sin^2\frac{1}{\sqrt{n}}}{1/n} = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{\sqrt{n}}}{1/\sqrt{n}}\right)^2 = \lim_{u\to 0}\left(\frac{\sin u}{u}\right)^2 = 1$, so this sum is divergent.

The same method comparison can be made for B). $\sin n$ does not converge to any value as n approaches infinity, so C) diverges by the n^{th} term test.

For D), the limit comparison test can be used with the Basel series $\frac{1}{n^2}$, which is known to be convergent. $\lim_{n\to\infty} \frac{\sin \frac{1}{n^2}}{1/n^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1$, so this sum is convergent.

16) The initial value is $\frac{1}{2}$, the common ratio is $\frac{1}{2}$, and there are 2021 terms. The sum is equal to $\frac{1/2(1-1/2^{2021})}{1-1/2} = 1 - \frac{1}{2^{2021}}$.

17) x is centered about $x = 0$ and the denominator has a power of 3, so the interval of convergence here contains $(-3, 3)$. Checking the endpoints, $x = -3$ results in the alternating harmonic series (which is convergent), and $x = 3$ results in the harmonic series (which is divergent). Thus, the interval of convergence is $[-3, 3)$.

18) $\sum_{n=1}^{1000} \left\lfloor \frac{2021n}{1001} \right\rfloor = \sum_{n=1}^{1000} \left\lfloor \frac{2021(1001-n)}{1001} \right\rfloor = \sum_{n=1}^{1000} \left(2021 + \left\lfloor \frac{-2021n}{1001} \right\rfloor \right)$. Since $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ for $x \notin \mathbb{Z}$, this is equal to $\sum_{n=1}^{1000} \left(2021 - 1 - \left\lfloor \frac{2021n}{1001} \right\rfloor \right) = 2020 \cdot 1000 - \sum_{n=1}^{1000} \left\lfloor \frac{2021n}{1001} \right\rfloor$. This is the original sum being subtracted, so the value of the sum is $2020 \cdot 1000/2 = 2^2 \cdot 5 \cdot 101 \times 2^3 \cdot 5^3/2 = 2^4 \cdot 5^4 \cdot 101$. This has $5 \cdot 5 \cdot 2 = 50$ positive integer factors.

19) Since x is a constant, the sum is equal to $-\frac{x^2 + x^4}{2 + x^2}$ for all real x. Since x is bounded, the sum is absolutely convergent.

20) I) can be true if $a(0) = -20$, $b(0) = 21$, and $a(x) = b(x) = 0$ otherwise. III) can be true since the sums only describe the functions at the discretized non-negative integer values of x, not any non-integer points. The definition of integrable includes countably infinite discontinuities, so the function described for I) would also satisfy IV). However, for II), this would cause $a(x)$ to tend towards $-\infty$ as x grows larger which contradicts its convergent sum.

21) $x^{\ln n} = n^{\ln x}$ when x is positive. The solution to $\ln x < -1$ is $x < \frac{1}{e}$. Negative numbers cannot be raised to irrational powers to produce real results, so the smallest value of x that produces a convergent sum is $x = 0$ and the interval of convergence is $\left[0, \frac{1}{e}\right)$.

22) Note that $\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$. For $\sum_{n=1}^{\infty} s_n$ to converge, s_n must tend to 0, meaning the sum of all values in $\{a_n\}$ is 0, so II) is true. However, if $\left| \sum_{n=1}^{\infty} a_n \right| > 0$, then $\lim_{n\to\infty} s_n = \delta > 0$

and $\sum_{n=1}^{\infty} s_n$ diverges, so I) is false.

The predicate of III) states that $\{a_n\}$ has terms of opposite signs. If all of the terms of $\{a_n\}$ are (WLOG) positive, then $\sum_{n=1}^{\infty} a_n > a_1$ and $\sum_{n=1}^{k} a_n = s_k > ka_1$. As k tends to infinity, s_k would approach $+\infty$ and diverge, a contradiction. Thus, III) must be true.

23) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e$, so $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ is divergent by the n^{th} term test. Checking the other choices produces inconclusive results.

24) $\int_1^{\infty}\frac{dx}{x^2\lfloor x\rfloor} = \sum_{n=1}^{\infty}\int_n^{n+1}\frac{dx}{nx^2} = \sum_{n=1}^{\infty}\frac{1}{n^3+n^2}$. By partial fractions, this is equal to $\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$. The first is the well-known Basel problem, which has a sum of $\frac{\pi^2}{6}$, and the second is a telescoping series whose only non-deleted term is 1. Thus, the integral has value $\frac{\pi^2}{6}-1$.

25) The ratio between successive terms in the Fibonacci sequence is $\frac{1+\sqrt{5}}{2}$, and it grows asymptotically to this. Thus, the radius of convergence of its generating function is $\frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$.

26) Since $\sqrt[n]{n}$ converges to 1 as n grows very large (which can be shown by taking a natural logarithm and applying l'Hospital's Rule), $\lim_{n\to\infty}(\sqrt[n]{n}-1) = 0$ and the series is absolutely convergent by the root test.

27) 1 is added $2021-(-2021)+1 = 4043$ times.

28) Using the Method of Finite Differences, we obtain the following.

$70-20n$		1		n		5		$2n$		15		$126-25n$
	$20n-69$		$n-1$		$5-n$		$2n-5$		$15-2n$		$111-25n$	
		$68-19n$		$6-2n$		$3n-10$		$20-4n$		$96-23n$		
			$17n-62$		$5n-16$		$30-7n$		$76-19n$			
				$46-12n$		$46-12n$		$46-12n$				

The only integer value of n that produces a value of $70-20n$ within 10 of 2021 is $n=-98$, producing 2030. $f(6) = 126-27n = 2576$

29) The denominator factorizes to $n(n+1)(n+2)(n+3)$, so let the fraction be equal to $\frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}+\frac{D}{n+3}$. Multiplying by the denominator, we obtain the equation $A(n+1)(n+2)(n+3)+Bn(n+2)(n+3)+Cn(n+1)(n+3)+Dn(n+1)(n+2) = n^2+3n+3$. Plugging in $n=0$ gives $6A=3$, plugging in $n=-1$ gives $-2B=1$, plugging in $n=-2$ gives $2C=1$, and plugging in $n=-3$ gives $-6D=3$. Thus, the sum is equal to

$\frac{1}{2}\sum_{n=1}^{48}\left(\frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}\right)$. This sum will telescope, with all but four terms canceling out. $\frac{1}{2}\left(1+\frac{1}{3}-\frac{1}{49}-\frac{1}{51}\right) = \frac{1616}{2499}$. $1616 + 2499 = 4115$, which leaves a remainder of 73 when divided by 2021.

30) Examples of series that create convergent sums are shown.

I) $a_n = -b_n = \frac{1}{n}$

II) $a_n = b_n = \frac{1}{n}$

III) $a_n = \frac{1}{n}$, $b_n = n$

IV) $c_n = 0$

V) $a_n = -\frac{1}{n}$, $b_n = n$, $c_n = \frac{1}{n^2}$

VI) Must diverge

VII) $a_n = n^2$

VIII) $a_n = \frac{1}{c_n} = n^2$

Only VI) must diverge.