Answer Key

0. 4079

1. 112

2. 628

3. 458

4. 54π

5. 47

6. 30

7. 257

8. -470

9. $\frac{2\sqrt{3}}{5}$

10. 32

11. -5

12. 2021

13. $23e$

14. 4043

Problem 0 Let $f(x) = x^3 + 2021x$

A. Let $A = f(1)$.

B. Let $B = f'(2)$.

C. Let $C = f''(3)$.

D. Let $D = f'''(4)$.

Evaluate $A + B + C + D$.

Solution 1 **Answer:** $\boxed{4079}$

A. $f(1) = 1^3 + 2021(1) = 2022$.

B. $f'(2) = (3)2^2 + 2021 = 2033$.

C. $f''(3) = (3)(2)(3) = 18$.

D. $f'''(4) = (3)(2) = 6$.

Putting everything together, $2022 + 2033 + 18 + 6 = \boxed{4079}$.

Problem 1 At time $t = 0$, Jae and Alex both stand at $(0,0)$ in the xy-plane. At that instant, Jae starts running along the positive x-axis such that his *position* at time t is given by $x_J(t) = 2t^2 + 4t$ and Alex starts running along the positive y-axis such that his *velocity* at time t is given by $v_A(t) = 6t$. Ignore units.

A. Let A be Jae's speed at time $t = 2$.

B. Let B be the total distance Alex travels between $t = 0$ and $t = 2$.

C. Let C be the unique nonzero value of t where Jae and Alex have travelled the same total distance.

D. Let D be the rate at which the distance between Alex and Jae is changing at time $t = 2$.

Evaluate $A + B + C + 5D$.

Solution 1 **Answer:** $\boxed{112}$

A. Taking the derivative, $v_J(t) = x_J'(t) = 4t + 4$. Plugging in $t = 2$ gives an answer of $\boxed{12}$.

B. Integrating, $y_A(t) = \int v_A(t)\,dt = 3t^2 + C$. Since $y_A(0) = 0$, $C = 0$ and $y_A(t) = 3t^2$. Plugging in $t = 2$ gives an answer of $\boxed{12}$.

C. From part B, we have that $y_A(t) = 3t^2$. Since Jae and Alex never change direction, we can simply set $y_A(t) = x_J(t) \iff 2t^2 + 4t = 3t^2 \rightarrow t = \boxed{4}$.

D. Let $d(t)$ denote the distance between Alex and Jae. By the distance formula (or Pythagorean theorem), $(d(t))^2 = (x_J(t))^2 + (y_A(t))^2$. Using implicit differentiation, we have that

$$2d(t)d'(t) = 2x_J(t)x_J'(t) + 2y_A y_A'(t) \iff d'(t) = \frac{x_J(t)v_J(t) + y_A(t)v_A(t)}{d(t)}$$

We can easily find that $x_J(2) = 16$ and $v_A(2) + 12$. From parts A and B, we have that $y_A(2) = 12$ and $v_J(2) = 12$. We can also find that $d(2) = \sqrt{12^2 + 16^2} = 20$. Plugging all of this in, we get

$$d'(2) = \frac{16(12) + 12(12)}{20} = \boxed{\frac{84}{5}}$$

Putting everything together, $12 + 12 + 4 + 5(\frac{84}{5}) = \boxed{112}$.

Problem 2

A. Let $A = \int_0^1 \left(x^{20} - x^{21}\right)\, dx$.

B. Let $B = \int_0^{\frac{\pi}{2}} \sin^{20}(x)\cos(x)\, dx$.

C. Let $C = \int_{20}^{21} \left(\frac{x - \lfloor x \rfloor}{\lfloor x \rfloor}\right) dx$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

D. Let $D = \int_0^1 \left(\frac{x}{4}\right)^2 \sqrt{1-x}\, dx$.

Evaluate $\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}$.

Solution 2 **Answer:** $\boxed{628}$

A. $\int_0^1 \left(x^{20} - x^{21}\right) dx = \left[\frac{1}{21}x^{21} - \frac{1}{22}x^{22}\right]_0^1 = \boxed{\frac{1}{462}}$.

B. Let $u = \sin(x)$. We have $du = \cos(x)\, dx$, so

$$\int_0^{\frac{\pi}{2}} \sin^{20}(x)\cos(x)\, dx = \int_0^1 u^{20}\, du = \left[\frac{1}{21}u^{21}\right]_0^1 = \boxed{\frac{1}{21}}.$$

C. Note that over the interval $(20, 21)$, $\lfloor x \rfloor$ always equals 20, so

$$\int_{20}^{21} \left(\frac{x - \lfloor x \rfloor}{\lfloor x \rfloor}\right) dx = \int_{20}^{21} \left(\frac{x - 20}{20}\right) dx = \frac{1}{20}\left[\frac{1}{2}(x-20)^2\right]_{20}^{21} = \boxed{\frac{1}{40}}.$$

D. Let $u = 1 - x$. We have $du = -dx$, so

$$\int_0^1 \left(\frac{x}{4}\right)^2 \sqrt{1-x}\, dx = -\frac{1}{16}\int_1^0 (1-u)^2\sqrt{u} = \frac{1}{16}\int_0^1 \left(u^{1/2} - 2u^{3/2} + u^{5/2}\right)$$
$$= \frac{1}{16}\left[\frac{2}{3}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right]_0^1 = \boxed{\frac{1}{105}}.$$

Putting everything together, $\frac{1}{\frac{1}{462}} + \frac{1}{\frac{1}{21}} + \frac{1}{\frac{1}{40}} + \frac{1}{\frac{1}{105}} = \boxed{628}$.

Problem 3 Functions f and g are both continuous and differentiable. Function f satisfies the property that $f(n) = 2n$ and $f'(n) = n$ for all integers n. Function g satisfies $g(1) = 20$ and $g'(1) = 21$.

A. Let $A = \frac{d}{dx}\left[f(g(x))\right]_{x=1}$

B. Let $B = \frac{d}{dx}\left[f(20x)f(21x)\right]_{x=1}$

C. Let $C = \frac{d}{dx}\left[\frac{(f(x))^2}{(g(x))^2}\right]_{x=1}$

D. Let $D_1 e^{D_2} = \frac{d}{dx}\left[e^{f(x)}e^{g(x)}\right]_{x=1}$, where D_1 and D_2 are integers.

Evaluate $A + \frac{B}{(20)(21)} + 20^3 \cdot C + D_1 + D_2$.

Solution 3 **Answer:** $\boxed{458}$

A. By the chain rule,

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x).$$

Plugging in $x = 1$ gives $f'(g(1))g'(1) = f'(20) \cdot 21 = 20(21) = \boxed{420}$

B. By the product and chain rules,

$$\frac{d}{dx}\left[f(20x)f(21x)\right] = f(20x) \cdot 21f'(21x) + 20f'(20x) \cdot f(21x).$$

Plugging in $x = 1$ gives $f(20) \cdot 21f'(21) + 20f'(20) \cdot 21f'(21) = 2(20)(21)(21) + 20(20)(2)(21) = 2(20)(21)(20+21) = (20)(21)(82)$. This is equal to $\boxed{34440}$, but the factored form is more convenient for the final compile.

C. Rewrite as $\frac{d}{dx}\left[\left(\frac{f(x)}{g(x)}\right)^2\right]$. By the quotient and chain rules,

$$\frac{d}{dx}\left[\left(\frac{f(x)}{g(x)}\right)^2\right] = 2\left(\frac{f(x)}{g(x)}\right)\left(\frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}\right).$$

Plugging in $x = 1$ gives $2(\frac{f(1)}{g(1)})(\frac{g(1)f'(1)-f(1)g'(1)}{g(1)^2}) = 2(\frac{2}{20})(\frac{20(1)-2(21)}{20^2}) = -\frac{88}{20^3}$. This can be simplified to $\boxed{-\frac{11}{1000}}$, but the unsimplified form is more convenient for the final compile.

D. Rewrite as $\frac{d}{dx}\left[e^{f(x)+g(x)}\right]$. By the chain rule,

$$\frac{d}{dx}\left[e^{f(x)+g(x)}\right] = e^{f(x)+g(x)}(f'(x) + g'(x)).$$

Plugging in $x = 1$ gives $e^{f(1)+g(1)}(f'(1) + g'(1)) = e^{2+20}(1 + 21) = 22e^{22}$, so $D_1 = \boxed{22}$ and $D_2 = \boxed{22}$.

Putting everything together, $420 + \frac{20(21)(82)}{20(21)} + 20^3(-\frac{88}{20^3}) + 22 + 22 = \boxed{458}$.

Problem 4

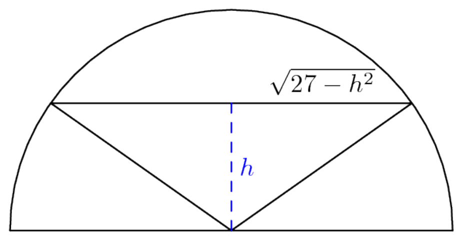
A. A right circular cone is inscribed "upside down" in a hemisphere of radius $3\sqrt{3}$ such that its base is parallel to the circular base of the hemisphere and its apex is at the center of the base of the hemisphere. Let A be the greatest possible volume of such a cone.

B. A right circular cylinder of base radius 2 and height 4 is inscribed in a right circular cone with the same axis. Let B be the least possible volume of such a cone.

Evaluate $A + B$.

Solution 4 **Answer:** $\boxed{54\pi}$

A. Consider a vertical cross-section of the hemisphere and cone that passes through a diameter of the hemisphere.



Let the cone have height h. By the Pythagorean theorem, the radius of the base of the cone is $\sqrt{27-h^2}$. The volume of the cone is given by

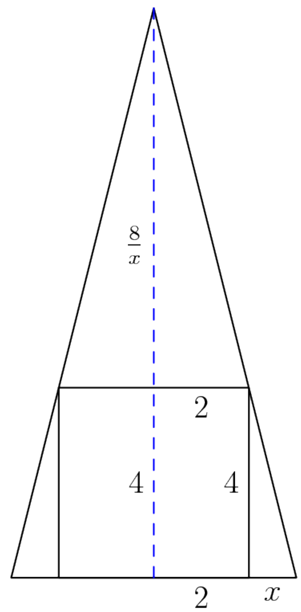
$$V = \frac{\pi}{3}(27-h^2)(h) = \frac{\pi}{3}(27h-h^3) \implies V' = \pi(9-h^2).$$

Setting $V' = 0$ gives $h = \pm 3$, of which only $h = 3$ makes sense. There are also critical points[1] at $h = 0$ and $h = 3\sqrt{3}$, but those clearly yield degenerate cases with volume 0. It's not hard to check (e.g. with the second derivative), that $h = 3$ is a local maximum. Plugging in $h = 3$ to our original expression gives $B = \frac{\pi}{3}(27-9)(3) = \boxed{18\pi}$.

(Solution continued on next page)

[1]When we found the expression for volume, we squared a square root and just canceled it. This is fine for all $0 < h < 3\sqrt{3}$, but if we consider the limit definition of the derivative with the square root present (just with a square on the outside), the left-sided limit does not exist for $h = 0$ and the right-sided limit does not exist for $h = 3\sqrt{3}$. Therefore, the derivative is undefined at these points.

B. Consider a vertical cross section that contains the shared axis of the cone and cylinder.



Let the radius of the base of the cone be $x+2$ for some x. Considering similar triangles gives the height of the cone as $4+\frac{8}{x}$. The volume of the cone is given by

$$V = \frac{\pi}{3}(2+x)^2\left(4+\frac{8}{x}\right) = \frac{4\pi}{3}\left(x^2+6x+12+\frac{8}{x}\right) \implies V' = \frac{8\pi}{3}\left(x+3-\frac{4}{x^2}\right)$$

Setting this equal to 0, we get $x^3+3x^2-4=(x-1)(x+2)^2=0$, so $x=1$ or $x=-2$. Only $x=1$ makes sense. There's also a critical point at $x=0$ (where V' is undefined), but V is not defined here, so we can ignore it. It's easy to verify (e.g. through second derivative), that $x=1$ yields a local minimum. Plugging in $x=1$ gives a volume of $\frac{\pi}{3}(1+2)^2\left(4+\frac{8}{1}\right) = \boxed{36\pi}$.

Putting everything together, $18\pi+36\pi=\boxed{54\pi}$.

Problem 5 The graph of the equation

$$y = x^6 + 2x^5 - 31x^4 - 64x^3 + 224x^2 + 532x + 277$$

is tangent to the line $y = mx + b$ at points (x_1, y_1), (x_2, y_2), and (x_3, y_3), where $x_1 < x_2 < x_3$.

Evaluate $m + b + x_1 + 2x_2 + 3x_3$.

Solution 5 **Answer:** $\boxed{47}$

For brevity, let $f(x) := x^6 + 2x^5 - 31x^4 - 64x^3 + 224x^2 + 532x + 277$. Note that the graph of $f(x) - (mx + b)$ must be tangent to the x-axis at $x = x_1, x_2, x_3$. Since a graph of a polynomial can only be tangent to the x-axis at a root of multiplicity greater than 1 and $f(x)$ has degree 6, we can write

$$f(x) = k(x - x_1)^2(x - x_2)^2(x - x_3)^2 + mx + b.$$

Since $f(x)$ has leading coefficient 1, clearly $k = 1$. Let $(x - x_1)(x - x_2)(x - x_3) = (x^3 + px^2 + qx + r)$ for real numbers p, q, r. We have that

$$x^6 + 2x^5 - 31x^4 - 64x^3 + 224x^2 + 532x + 277 = (x^3 + px^2 + qx + r)^2 + mx + b.$$

Expanding the right-hand side gives

$$x^6 + 2px^5 + (p^2 + 2q)x^4 + (2r + 2pq)x^3 + (2pr + q^2)x^2 + (2qr + m)x + (r^2 + b).$$

Matching coefficients yields the equations

$$\begin{aligned} 2p &= 2 \\ p^2 + 2q &= -31 \\ 2r + 2pq &= -64 \\ 2pr + q^2 &= 224 \\ 2qr + m &= 532 \\ r^2 + b &= 277 \end{aligned}$$

Solving these equations[2] gives $p = 1$, $q = -16$, $r = -16$, $\boxed{m = 20}$, and $\boxed{b = 21}$. This means we can write

$$(x - x_1)(x - x_2)(x - x_3) = x^3 + x^2 - 16x - 16 = (x + 4)(x + 1)(x - 4),$$

so $\boxed{x_1 = -4}$, $\boxed{x_2 = -1}$, and $\boxed{x_3 = 4}$.

Putting everything together, $20 + 21 + (-4) + 2(-1) + 3(4) = \boxed{47}$.

[2] Although this looks bad, starting from the first, each equation introduces only one new variable, so it works out fairly cleanly.

Problem 6

A. Let $A = \lim_{x\to 20}\left[\dfrac{x^2 - 41x + 420}{x - 20}\right]$

B. Let $B = \lim_{x\to -\infty}\left[\dfrac{20e^x + 21e^{-x}}{2e^x + e^{-x}}\right]$

C. Let $C = \lim_{x\to 0}\left[\dfrac{\sin(x) - x\cos(x)}{x^3}\right]$

D. Let $D = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\dfrac{1}{\sqrt{i\cdot n}}\right]$

Evaluate $A + B + 24C + D$.

Solution 6 **Answer:** $\boxed{30}$

A. Factoring the numerator,

$$\lim_{x\to 20}\left[\frac{x^2 - 41x + 420}{x - 20}\right] = \lim_{x\to 20}\left[\frac{(x-20)(x-21)}{x-20}\right] = \lim_{x\to 20}(x - 21) = \boxed{-1}.$$

B. Note that $e^x \to 0$ as $x \to -\infty$ (and e^{-x} does not), so

$$\lim_{x\to -\infty}\left[\frac{20e^x + 21e^{-x}}{2e^x + e^{-x}}\right] = \lim_{x\to -\infty}\left[\frac{21e^{-x}}{e^{-x}}\right] = \boxed{21}.$$

C. A single application of L'Hôpital's Rule gives

$$\lim_{x\to 0}\left[\frac{\sin(x) - x\cos(x)}{x^3}\right] = \lim_{x\to 0}\left[\frac{x\sin(x)}{3x^2}\right] = \lim_{x\to 0}\left[\frac{\sin(x)}{3x}\right] = \boxed{\frac{1}{3}}.$$

D. This setup with a limit and a sum resembles the setup for a right-handed Riemann sum with n equal subintervals over the interval $(0, 1)$. Indeed, if we rearrange the summand a bit, we can find that

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{\sqrt{i\cdot n}}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\sqrt{\frac{n}{i}}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{\frac{i}{n}}}\right],$$

which is indeed the left-handed Riemann sum approximation for $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ with n equal subintervals. This means that

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{\frac{i}{n}}}\right] = \int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_0^1 = \boxed{2}.$$

Putting everything together, $-1 + 21 + 24(\frac{1}{3}) + 2 = \boxed{30}$.

Problem 7 Let $\mathcal{R}$ be the finite region bounded by the graphs of $y = x$ and $y = 2x - x^2$.

A. Let A be the area of $\mathcal{R}$.

B. If $\mathcal{R}$ is revolved about the x-axis, let B be the volume of the solid that results.

C. If $\mathcal{R}$ is revolved about the y-axis, let C be the volume of the solid that results.

D. If a solid has $\mathcal{R}$ as its base, and cross sections of the solid taken perpendicular to the x-axis are semicircles with diameters in $\mathcal{R}$, let D be the volume of such a solid.

Evaluate $\dfrac{1}{A} + \dfrac{\pi}{B} + \dfrac{\pi}{C} + \dfrac{\pi}{D}$.

Solution 7 **Answer:** $\boxed{257}$

Note that the two curves intersect at $x = 0$ and $x = 1$ and that $2x - x^2 > x$ over $0 < x < 1$.

A. The area is given by

$$\int_0^1 \left[(2x - x^2) - x\right] dx = \left[\tfrac{1}{2}x - \tfrac{1}{3}x^2\right]_0^1 = \boxed{\frac{1}{6}}.$$

B. Using the washer method, the volume is given by

$$\pi \int_0^1 \left[(2x - x^2)^2 - (x)^2\right] dx = \pi \int_0^1 \left[3x^2 - 4x^3 + x^4\right] dx = \pi \left[x^3 - x^4 + \tfrac{1}{5}x^5\right]_0^1 = \boxed{\frac{\pi}{5}}.$$

C. Using the shell method, the volume is given by

$$2\pi \int_0^1 \left[x((2x - x^2) - x)\right] dx = 2\pi \int_0^1 \left[x^2 - x^3\right] dx = 2\pi \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 = \boxed{\frac{\pi}{6}}.$$

D. The area of a semicircle with diameter d is $\frac{\pi}{8}d^2$, so the volume is given by

$$\frac{\pi}{8} \int_0^1 \left[((2x - x^2) - x)^2\right] dx = \frac{\pi}{8} \int_0^1 \left[x^2 - 2x^3 + x^4\right] dx = \frac{\pi}{8} \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5\right]_0^1 = \boxed{\frac{\pi}{240}}.$$

Putting everything together, $\dfrac{1}{\frac{1}{6}} + \dfrac{\pi}{\frac{\pi}{5}} + \dfrac{\pi}{\frac{\pi}{6}} + \dfrac{\pi}{\frac{\pi}{240}} = \boxed{257}$

Problem 8

A. Let $A = \int_0^1 x^{21} \ln(x)\, dx$.

B. Let $B = \int_0^\infty e^{-x} \cos(2x)\, dx$.

C. Let $C = \int_0^\infty e^{-x} \cos^2(x)\, dx$.

D. Let $D = \int_0^\infty e^{-x} \sin^2(x)\, dx$.

Evaluate $\frac{1}{A} + 25(B^2 + C^2 + D^2)$.

Solution 8 **Answer:** $\boxed{-470}$

A. Integrate by parts with $u = \ln(x)$ and $dv = x^{21} dx$.

$$\int_0^1 x^{21} \ln(x)\, dx = \left[\tfrac{1}{22} x^{22} \ln(x)\right]_0^1 - \int_0^1 \tfrac{1}{22} x^{22} \cdot \tfrac{1}{x}\, dx = \left[\tfrac{1}{22} x^{22} \ln(x)\right]_0^1 - \left[\tfrac{1}{484} x^{22}\right]_0^1 = \boxed{-\frac{1}{484}}$$

B. Let $B = \int_0^\infty e^{-x} \cos(2x)\, dx$. Integrate by parts with $u = \cos(2x)$ and $dv = e^{-x}\, dx$.

$$\int_0^\infty e^{-x} \cos(2x)\, dx = \left[-e^{-x} \cos(2x)\right]_0^\infty - 2\int_0^\infty e^{-x} \sin(2x)\, dx = 1 - 2\int_0^\infty e^{-x} \sin(2x)\, dx$$

To evaluate the latter integral, integrate by parts again with $u = \sin(2x)$ and $dv = e^{-x}\, dx$.

$$\int_0^\infty e^{-x} \sin(2x)\, dx = \left[-e^{-x} \sin(2x)\right]_0^\infty + 2\int_0^\infty e^{-x} \cos(2x)\, dx = 2B.$$

Substituting back into the first equation, we find that

$$B = 1 - 2(2B) \implies B = \boxed{\frac{1}{5}}$$

C. Let $C = \int_0^\infty e^{-x} \cos^2(x)\, dx$ and $D = \int_0^\infty e^{-x} \sin^2(x)\, dx$. We have that

$$C + D = \int_0^\infty e^{-x} \left(\cos^2(x) + \sin^2(x)\right)\, dx = \int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1$$
$$C - D = \int_0^\infty e^{-x} \left(\cos^2(x) - \sin^2(x)\right)\, dx = \int_0^\infty e^{-x} \cos(2x)\, dx = \frac{1}{5}$$

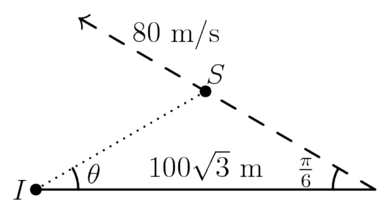
Solving the system, we find that $C = \boxed{\frac{3}{5}}$ and $D = \frac{2}{5}$.

D. See the solution to part C. $D = \boxed{\frac{2}{5}}$.

Putting everything together, $\frac{1}{-\frac{1}{484}} + 25\left((\tfrac{1}{5})^2 + (\tfrac{3}{5})^2 + (\tfrac{2}{5})^2\right) = \boxed{-470}$

Problem 9 Shreyas, ace pilot of the American Heritage math team, is waiting on the (perfectly flat) runway at American Heritage International Airport. Iris, perhaps unwisely, is lying down on the other end of the runway (such that her eye level is exactly at the ground), facing Shreyas in his plane. Ignore the dimensions of Shreyas's plane.

Shreyas accelerates to a constant 80 meters per second (moving directly towards Iris) and takes off at an angle of $\frac{\pi}{6}$ radians to the ground at the instant he is $100\sqrt{3}$ meters away from Iris. Iris keeps looking directly at Shreyas's plane, and after 1.25 seconds, her angle of elevation is $\theta = \frac{\pi}{6}$ and is increasing at a rate of θ' radians per second. Compute θ'.



Solution 9 **Answer:** $\boxed{\frac{2\sqrt{3}}{5}}$

Establish a coordinate system with Iris at the origin. Let $t = 0$ be the instant Shreyas's plane takes off at the point $(100\sqrt{3}, 0)$. Decompose Shreyas's velocity vector into $v = \langle -80\cos(\frac{\pi}{6}), 80\sin(\frac{\pi}{6}\rangle = \langle -40\sqrt{3}, 40\rangle$. Therefore, Shreyas's position at time t is given by

$$S(t) = (100\sqrt{3} - 40t\sqrt{3}, 40t).$$

Now, note that Iris's angle of elevation θ satisfies

$$\tan(\theta) = \frac{S_y(t)}{S_x(t)} = \frac{40t}{100\sqrt{3} - 40t\sqrt{3}}.$$

Rearranging and simplifying, we can see that this is equivalent to

$$\sqrt{3}\tan(\theta) = \frac{2t}{5-2t} = -1 + \frac{5}{5-2t}.$$

Differentiating with respect to t, we find that

$$\sqrt{3}\sec^2(\theta)\theta' = \frac{10}{(5-2t)^2} \iff \theta' = \frac{10\cos^2(\theta)}{\sqrt{3}(5-2t)^2}.$$

Plugging in $\theta = \frac{\pi}{6}$ and $t = 1.25$, we get that $\theta' = \frac{10(\frac{3}{4})}{\sqrt{3}(\frac{5}{2})^2} = \boxed{\frac{2\sqrt{3}}{5}}$.

Problem 10

A. Let $A = \int_0^2 \frac{x^7}{1+x+x^2+x^3}\,dx$

B. Let $B = \int_0^2 \frac{x^8}{1+x+x^2+x^3}\,dx$

C. Let $C = \int_0^2 \frac{x^9}{1+x+x^2+x^3}\,dx$

D. Let $D = \int_0^2 \frac{x^{10}}{1+x+x^2+x^3}\,dx$

Evaluate $A+B+C+D$.

Solution 10 **Answer:** $\boxed{32}$

These integrals can be evaluated separately, but that would be a rather unpleasant experience. Fortunately, one can note that

$$A+B+C+D = \int_0^2 \frac{x^7+x^8+x^9+x^{10}}{1+x+x^2+x^3}\,dx = \int_0^2 \frac{x^7(1+x+x^2+x^3)}{1+x+x^2+x^3}\,dx = \int_0^2 x^7\,dx = \left[\tfrac{1}{8}x^8\right]_0^2 = \boxed{32}.$$

If you're curious:

$$\int_0^2 \frac{x^7}{1+x+x^2+x^3}\,dx = \frac{22}{5} - \frac{\ln(3)}{2} - \frac{\ln(5)}{4} - \frac{\arctan(2)}{2}$$
$$\int_0^2 \frac{x^8}{1+x+x^2+x^3}\,dx = \frac{64}{15} + \frac{\ln(3)}{2} - \frac{\ln(5)}{4} + \frac{\arctan(2)}{2}$$
$$\int_0^2 \frac{x^9}{1+x+x^2+x^3}\,dx = \frac{58}{7} - \frac{\ln(3)}{2} + \frac{\ln(5)}{4} + \frac{\arctan(2)}{2}$$
$$\int_0^2 \frac{x^{10}}{1+x+x^2+x^3}\,dx = \frac{316}{21} + \frac{\ln(3)}{2} + \frac{\ln(5)}{4} - \frac{\arctan(2)}{2}$$

which do, indeed, sum to 32.

Problem 11 Let $\mathcal{L}$ be the set of all lines of the form $x = k$ or $y = k$ for any integer k.

A. If a circular disk of radius $\frac{1}{4}$ is randomly dropped onto the coordinate plane, let A be the probability that it does not intersect any of the lines in $\mathcal{L}$.

B. If, instead of a circular disk, a square tile with circumradius $\frac{1}{2}$ is randomly dropped onto the coordinate plane (such that both its position and orientation are random), the probability it does not cross any of the lines in $\mathcal{L}$ can be expressed as

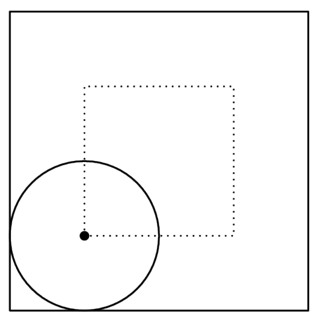
$$B + \frac{1}{\pi}(C + D\sqrt{2})$$

for (not necessarily positive) rational numbers B, C, and D.

Evaluate $4(A + B + C + D)$.

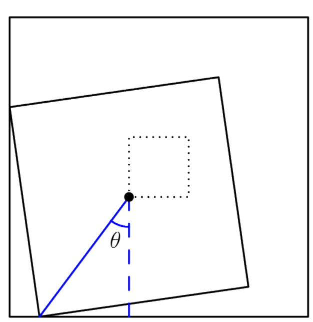
Solution 11 **Answer:** $\boxed{-5}$

A. Note that $\mathcal{L}$ partitions the plane into a tiling of unit squares. We only need to consider one such square, as the entire tiling is symmetric.



The radius of $\frac{1}{4}$ creates a "border" of width $\frac{1}{4}$. If the circle's center were to land inside this border, the circle would cross one of the outer edges of the square, which is a line in $\mathcal{L}$. Therefore, the set of all points that the circle's center could land on is the smaller square that remains after this border is removed, which has side length $1 - 2(\frac{1}{4}) = \frac{1}{2}$. This corresponds to an area of $\frac{1}{4}$, and since the area of the total square is 1, our desired probability is simply $\boxed{\frac{1}{4}}$.

B. Again, consider a single unit square. Suppose the tile lands such that its diagonals are displaced by an angle θ from the coordinate axes (where $0 \leq \theta \leq \frac{\pi}{4}$).



(Solution continued on next page)

The tile creates a "border" of width $\frac{1}{2}\cos(\theta)$. Using a similar process to part A, this means that, for a fixed angle θ, the set of all points that the square's center could land on is a square with side length $1 - 2(\frac{1}{2}\cos(\theta)) = 1 - \cos(\theta)$, which has area $(1 - \cos(\theta))^2$. The desired probability is therefore the expected value of this area as θ ranges from 0 to $\frac{\pi}{4}$.[3] This is given by

$$\frac{4}{\pi}\int_0^{\frac{\pi}{4}} (1 - \cos(\theta))^2\, d\theta = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} (1 - 2\cos(\theta) + \cos^2(\theta))\, d\theta = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} \left(1 - 2\cos(\theta) + \frac{1 + \cos(2\theta)}{2}\right) d\theta$$

Where the last equality comes from the double angle identity for cosine. Finishing, this is equal to

$$\frac{4}{\pi}\left[\frac{3\theta}{2} - 2\sin(\theta) + \frac{\sin(2\theta)}{4}\right]_0^{\frac{\pi}{4}} = \frac{3}{2} + \frac{1}{\pi}\left(1 - 4\sqrt{2}\right),$$

so $\boxed{B = \frac{3}{2}}$, $\boxed{C = 1}$, and $\boxed{D = -4}$.

Putting everything together, $4(\frac{1}{4} + \frac{3}{2} + 1 - 4) = \boxed{-5}$.

[3]Just as before, the denominator in our geometric probability is the area of the total square, which is 1. The reason we let θ range only to $\frac{\pi}{4}$ instead of 2π is because the location of the reference angle θ changes every $\frac{\pi}{4}$ radians.

Problem 12

A. Let $A = \lim_{x\to 0^+}\left[\frac{\ln(x)}{\ln(\sqrt{x})}\right]$

B. Let $B = \int_{-1}^{1} \frac{\arcsin(x^3)}{x^{20}+21}\,dx$

C. Let $C = \sum_{n=1}^{\infty} \frac{d^n}{dx^n}\left[e^{2x/3}\right]_{x=0}$

D. Let $D = \frac{d}{dx}\left[\int_0^{x^2} \frac{1}{1+t^2}\,dt\right]_{x=1}$

Evaluate $1000A + 100B + 10C + D$.

Solution 12 **Answer:** $\boxed{2021}$

A. Note that for all $x > 0$,

$$\frac{\ln(x)}{\ln(\sqrt{x})} = \frac{\ln(x)}{\frac{1}{2}\ln(x)} = 2,$$

so our desired limit is $\boxed{2}$.

B. The integral of an odd function over symmetric bounds is $\boxed{0}$.

C. The nth derivative is given by

$$\frac{d^n}{dx^n}\left[e^{2x/3}\right]_{x=0} = \left[\left(\frac{2}{3}\right)^n e^{2x/3}\right]_{x=0} = \left(\frac{2}{3}\right)^n.$$

This gives a simple geometric series:

$$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \boxed{2}.$$

D. By the first fundamental theorem of calculus and the chain rule,

$$\frac{d}{dx}\left[\int_0^{x^2}\frac{1}{1+t^2}\,dt\right]_{x=1} = \left[\frac{1}{1+x^4}\cdot 2x\right]_{x=1} = \boxed{1}.$$

Putting everything together, $1000(2) + 100(0) + 10(2) + 1 = \boxed{2021}$.

Problem 13

A. Let $A = \sum_{n=0}^{\infty} \frac{n}{n!}$.

B. Let $B = \sum_{n=0}^{\infty} \frac{n^2}{n!}$.

C. Let $C = \sum_{n=0}^{\infty} \frac{n^3}{n!}$.

D. Let $D = \sum_{n=0}^{\infty} \frac{n^4}{n!}$.

Evaluate $A + B + C + D$.

Solution 13 **Answer:** $\boxed{23e}$

For brevity, let $S_p := \sum_{n=0}^{\infty} \frac{n^p}{n!}$. Note that $S_0 = e$ by the Taylor series definition of e^x.

A. Manipulating, we find that

$$S_1 = \sum_{n=0}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = S_0 = \boxed{e}.$$

B. Manipulating, we find that

$$S_2 = \sum_{n=0}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=1}^{\infty} \left(\frac{n-1}{(n-1)!} + \frac{1}{(n-1)!} \right) = S_1 + S_0 = \boxed{2e}.$$

C. Manipulating, we find that

$$S_3 = \sum_{n=0}^{\infty} \frac{n^3}{n!} = \sum_{n=1}^{\infty} \frac{n^2}{(n-1)!} = \sum_{n=1}^{\infty} \left(\frac{(n-1)^2}{(n-1)!} + \frac{2(n-1)}{(n-1)!} + \frac{1}{(n-1)!} \right) = S_2 + 2S_1 + 1 = \boxed{5e}.$$

D. Manipulating, we find that

$$\begin{aligned} S_4 &= \sum_{n=0}^{\infty} \frac{n^4}{n!} = \sum_{n=1}^{\infty} \frac{n^3}{(n-1)!} = \sum_{n=1}^{\infty} \left(\frac{(n-1)^3}{(n-1)!} + \frac{3(n-1)^2}{(n-1)!} + \frac{3(n-1)}{(n-1)!} + \frac{1}{(n-1)!} \right) \\ &= S_3 + 3S_2 + 3S_1 + S_0 = \boxed{15e}. \end{aligned}$$

Putting everything together, $e + 2e + 5e + 15e = \boxed{23e}$.

Problem 14 Consider the following function, where the set contains each term of the Maclauren series expansion of e^x.

$$\mathcal{F}(x) = \max\left\{1, x, \frac{x^2}{2}, \frac{x^3}{6}, \frac{x^4}{24}, \dots, \frac{x^n}{n!}, \dots\right\}$$

The interval of all real numbers r such that $\mathcal{F}(r) = \frac{r^{2021}}{2021!}$ is $[m, n]$, for real numbers m and n. Evaluate $m + n$.

Solution 14 **Answer:** $\boxed{4043}$

Comparing consecutive terms, we can make the following two key observations:

- $\dfrac{x^k}{k!} < \dfrac{x^{k-1}}{(k-1)!}$ for all $x < k$. (1)
- $\dfrac{x^k}{k!} < \dfrac{x^{k+1}}{(k+1)!}$ for all $x > k+1$. (2)

For brevity, let $f_k(x) = \frac{x^k}{k!}$. Our desired condition becomes $\mathcal{F}(r) = f_{2021}(r)$. From (1), we can see that $\mathcal{F}(r) > f_{2021}(r)$ for all $r < 2021$ (as there exists at least one $k < 2021$ such that $f_{2021}(r) < f_k(r)$. From (2), we can see that $\mathcal{F}(r) > f_{2021}(r)$ for all $r > 2022$.

We claim that all remaining r, $r \in [2021, 2022]$, satisfy $\mathcal{F}(r) = f_{2021}(r)$. Indeed, temporarily ignoring the endpoints, repeated applications of (2) show that $f_{2021}(r) > f_k(r)$ for all $k < 2021$, and repeated applciations of (1) show that $f_{2021}(r) > f_k(r)$ for all $k \geq 2022$, so $f_{2021}(r)$ would be the maximum called by $\mathcal{F}$. Similar reasoning gives $\mathcal{F}(2021) = f_{2021}(2021) = f_{2020}(2021)$ and $\mathcal{F}(2022) = f_{2021}(2022) = f_{2022}(2022)$, so the endpoints also work.

This gives a final answer of $2021 + 2022 = \boxed{4043}$.