NOTA denotes "None of the Above." All variables are elements of $\mathbb{R}$ unless otherwise stated.

1. Evaluate $\displaystyle\int_1^e \frac{(\ln x)^{2021}}{x}\,dx$.

(A) $\frac{1}{2020}$ (B) $\frac{1}{2021}$ (C) $\frac{1}{2022}$ (D) $\frac{e}{2021}$ (E) NOTA

2. Evaluate $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{(1-x^n)^3}{(1+x^3)^n}\,dx$.

(A) 0 (B) 1 (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{3}$ (E) NOTA

3. Andrew is rather particular about the type of cake he is willing to eat on his birthday, and will only eat cakes formed by very peculiar constructions. Fortunately, you know just the shape of cake Andrew loves! The base of Andrew's dream cake is the triangle determined by the points $(-4,3),(4,11),(0,-1)$ on the Cartesian plane. The cake is constructed by forming regular octagonal cross sections perpendicular to the x-axis. Find the volume of Andrew's dream cake.

(A) $\frac{512(1+\sqrt{2})}{3}$ (B) $\frac{1024(1+\sqrt{2})}{3}$ (C) $\frac{2048(1+\sqrt{2})}{3}$ (D) $\frac{4096(1+\sqrt{2})}{3}$ (E) NOTA

4. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 4x}{\sin x}\,dx$.

(A) $\frac{3}{5}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{4}{3}$ (E) NOTA

5. Jackson the alien has the unique ability that he can fly to any height above the surface of a spherical planet similar to Earth with a radius of exactly 4000 miles. He utilizes this ability to maximize how much of the surface of the planet he can see at any given instant. He can see infinitely far in any direction. Given that he flies to a height of 100 miles radially away from the surface of planet, he can see exactly $k\%$ its total surface area. Which of the following is closest to k?

(A) 1.0 (B) 1.1 (C) 1.2 (D) 1.3 (E) NOTA

Questions 6-7 refer to the following information:
Couper is absolutely famished after his workout, and he has just finished his 1000^{th} daily workout. To celebrate this Herculean achievement, he has set out to create a cake which resembles his favorite curve, the cycloid. He begins by placing a circle of radius 1 and center $(0,1)$ on the Cartesian plane and he tracks the point on the circle currently at the origin, A. He then rolls this circle along the positive x-axis, tracing out the curve made by the path of A until A is at the coordinate $(\pi,0)$. Couper then revolves the resulting curve to create the infinitesimally thin mold for his dream cake.

6. What is the volume of Couper's cake mold?

(A) $2\pi^2$ (B) $\frac{5\pi^2}{2}$ (C) $3\pi^2$ (D) $\frac{7\pi^2}{2}$ (E) NOTA

7. What is the surface area of Couper's cake mold?

(A) $\frac{32\pi}{5}$ (B) $\frac{64\pi}{5}$ (C) $\frac{32\pi}{3}$ (D) $\frac{64\pi}{3}$ (E) NOTA

8. Jack and his son Danger are playing a game called Blitz! wherein Jack rolls a standard 7-sided die (Jack and Danger live in a magical world in which such a die indisputably exists), and Danger rolls a standard 4-sided die. In the first round of the game, Jack and Danger both roll their dice once. In subsequent rounds, they roll their dice the number of times equal to the last value they rolled in the previous round. A player's score for the game is the sum of all values shown by the dice in all rounds, and the winner is the player with more points at the end of the game. However, Danger soon realizes that the game is not in his favor and claims that he should be able to roll his dice k more times in addition to the last value he rolled in the previous round (in each round after the first) in order for the game to be fair. For such an integer k to exist, how many rounds should be played per game?

(A) 7 (B) 8 (C) 9 (D) 10 (E) NOTA

9. Triangle ABC is changing shape in time, where AB is growing at a rate of 3 meters per second, AC is shrinking at a rate of 1 meter per second, and angle A is growing at a rate of $\frac{\pi}{4}$ radians per second. At time $t = 0$, where t is measured in seconds, $AB = 3$, $AC = 4$ and $\angle A = \frac{\pi}{2}$. Let the rate at which side BC is changing at time $t = 1$ be k. Given that k can be written in the form $\sqrt{a + b\sqrt{c}} + d\pi$, where a, b and c are positive, square-free integers and d is an algebraic number, find $a + b + c$.

(A) 6 (B) 7 (C) 8 (D) 9 (E) NOTA

10. Sophie has become infatuated with clever factorizations, and wishes to test out her skills on as many integrals as possible. Help her get started by computing the value of $\int_0^1 \frac{1}{x^4 + 4} dx$.

(A) $\frac{1}{16}\left(\ln 5 + \frac{\pi}{4}\right)$
(B) $\frac{1}{32}(\ln 5 + \arctan 2)$
(C) $\frac{1}{16}(\ln 5 + \arctan 2)$
(D) $\frac{1}{8}(\ln 3 + \arctan 2)$
(E) NOTA

11. Sameera is obsessed with finding limits, particularly those which appear to diverge. Help her evaluate the following limit: $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$.

(A) $e^{-1/3}$ (B) $e^{-1/4}$ (C) 0 (D) DNE (E) NOTA

12. Gabriel's favorite number G and Jim's favorite number J form the unique ordered pair of nonzero real coordinates (G, J) such that $\lim_{x \to 0} \frac{x - \sin(x + Gx^3)}{x^5}$ exists and is equal to J. Find $(GJ)^{-1}$.

(A) 80 (B) 96 (C) 108 (D) 120 (E) NOTA

13. Jessica and June have been tasked with computing a rather interesting limit, however their blundering professor gave them the value of the limit rather than the full question! Realizing his grave mistake, he attempts to rectify the situation by having them find values of a and b such that the following limit holds:

$$\lim_{x \to 0} \left(\frac{\sin 3x}{x^3} + \frac{a}{x^2} + b\right) = 0$$

Find $a + b$.

(A) $-\frac{3}{2}$ (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{2}$ (E) NOTA

Questions 14-17 all refer to the largest finite region $\mathcal{R}$ bounded by the functions $f(x) = -x^2 + 8x - 12$, $g(x) = 2x - 3$, and the x-axis.

14. What is the area of $\mathcal{R}$?

(A) $\frac{45}{4}$ (B) $\frac{45}{2}$ (C) $\frac{16}{3}$ (D) $\frac{32}{3}$ (E) NOTA

15. What is the volume of the solid formed by revolving $\mathcal{R}$ about the x-axis?

(A) $\frac{6309\pi}{160}$ (B) $\frac{512\pi}{15}$ (C) $\frac{5421\pi}{160}$ (D) $\frac{351\pi}{10}$ (E) NOTA

16. What is the volume of the solid formed by revolving $\mathcal{R}$ about the y-axis?

(A) $\frac{351\pi}{4}$ (B) $\frac{2673\pi}{32}$ (C) $\frac{128\pi}{3}$ (D) $\frac{256\pi}{3}$ (E) NOTA

17. The line $x = a$ splits $\mathcal{R}$ into two regions of equal area. What interval must a lie within?

(A) $(2, 3)$ (B) $(3, 4)$ (C) $(4, 5)$ (D) $(5, 6)$ (E) NOTA

Questions 18-21 refer to the following information:

Soryan is standing atop a tall cliff of height 100 meters. He launches a rather weighty, 50 kg projectile off of the cliff at an angle of 60 degrees from the horizontal. Assume the acceleration due to gravity is 10 meters/sec^2.

18. The projectile takes a brisk 10 seconds to land on the ground below. What speed did Soryan launch the projectile, in meters per second?

(A) $\frac{40\sqrt{3}}{3}$ (B) $\frac{80\sqrt{3}}{3}$ (C) $\frac{40}{3}$ (D) $\frac{80}{3}$ (E) NOTA

19. How far away from the foot of the cliff does the weighty projectile land, in meters?

(A) $\frac{100}{3}$ (B) $\frac{200}{3}$ (C) $\frac{200\sqrt{3}}{3}$ (D) $\frac{400\sqrt{3}}{3}$ (E) NOTA

20. During this projectile's long, arduous, plummeting descent down to the cold, hard earth, what is the peak height, in meters, the projectile attains?

(A) $\frac{970}{9}$ (B) 105 (C) 110 (D) $\frac{980}{9}$ (E) NOTA

21. Tyger is using his brute force and sheer willpower to lug a large chain up the aforementioned cliff, however, a pesky monkey has fixed itself to the bottom end of the chain! The chain has uniform density, weighing 10 kg/meter, and the monkey suffers from a rare disease in which he loses mass at a rate directly proportional to how high up he has been dragged by Tyger. By the time the monkey reaches the top of the cliff, he will have exactly half of his current mass, which is 20 kg. How much work, in kilojoules, must Tyger exert to lug this monstrous chain and pesky monkey?

(A) 515 (B) 530 (C) 545 (D) 560 (E) NOTA

22. Felipe and Donovan, in an effort to combat the growing fear of secret societies such as the Illuminati, have begun a journey to find as many eligible members to join their own secret society, the Council of Maxwell. They must first create a distinguishable logo by inscribing the largest possible triangle into an ellipse with semi-major axis 8 and semi-minor axis 4. What is the area of this triangle?

(A) $24\sqrt{3}$ (B) $48\sqrt{3}$ (C) 12π (D) 24π (E) NOTA

23. Couper is obsessed with the physics of his running routines, and subsequently knows every minute detail about his training regime. One day, Couper is running on a track back and forth, with his position from the starting point modeled by the function $x(t) = \sin^4(t) + \cos^4(t)$. What is the jerk of Couper's position at time $t = \frac{5\pi}{12}$?

(A) $-4\sqrt{3}$ (B) $8\sqrt{3}$ (C) $4\sqrt{3}$ (D) $-8\sqrt{3}$ (E) NOTA

24. Jim is running a very profitable business in which his company's net worth is growing according to the function $P(t) = t^{\frac{5}{2}} + 3t$, where t is measured in years since the company's creation. Using the tangent line to $P(t)$ at $t = 4$, estimate the net worth of Jim's company after running for 5 years.

(A) 58 (B) 64 (C) 67 (D) 82 (E) NOTA

25. Jackson has recently discovered the secret to Iron Man's flight technology, however only within the two-dimensional plane. Nevertheless, he decides to test the limits of this technology and take it for a spin. Given that his position in the Cartesian plane can be described parametrically by the equations $x = t\cos t^2$ and $y(t) = t^2\cos t$, what is Jackson's speed at $t = 1$?

(A) $\sqrt{3-2\cos 1}$ (C) $\sqrt{5-2\sin 1}$ (E) NOTA
(B) $\sqrt{7-4\cos 2}$ (D) $\sqrt{5-4\sin 2}$

26. Andrew is fed up with using simple methods to solve for roots of his polynomials, instead opting to learn more advanced, computational methods, such as Newton's method. His favorite point to start with is $x_0 = \pi$, regardless of the function. In particular, he is interested in the function $f(x) = x^2 - 4x + 4$. Let the positive difference between Andrew's approximation of the root upon the third iteration of Newton's method from the actual root of $f(x)$ be a. What is $\lfloor 64a \rfloor$?

(A) 8 (B) 9 (C) 10 (D) 12 (E) NOTA

27. Vlad initiates a rampage against Saaketh's favorite spherical globe of the Earth with radius 1 meter. Vlad decides to start at the North Pole and head to the 30^{th} parallel, at which he bores a circular hole through the globe. What volume of the globe has Vlad removed, in cubic meters? (The 30^{th} parallel runs parallel to the equator and is 30 degrees north and south of the equator). Circumferences of this hole include the 30^{th} parallels of the Earth.

(A) $\frac{5\pi}{12}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{6}$ (D) $\frac{7\pi}{6}$ (E) NOTA

28. In anticipation of a cold, frigid winter during his college years, Shayaan has become increasingly fond of snow, particularly the beautiful, one-of-a-kind snowflakes one rarely sees. He has become infatuated with one particular figure, the Koch snowflake. Through extensive research on Wikipedia, Shayaan arrived at the following recursive definition for its construction:

> Start with an equilateral triangle of side length s_0, and recursively alter each line segment as follows.
>
> (1) divide the line segment into three segments of equal length.
>
> (2) draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
>
> (3) remove the line segment that is the base of the triangle from step 2.
>
> After n iterations, the area of the Koch snowflake is A_n and the perimeter is P_n.

Shayaan has set his sights on discovering this snowflake in nature, but must first compute the following: $\lim_{n\to\infty} A_n$, given $s_0 = 3^{1/4}$. What is this value?

(A) $\frac{3}{4}$ (B) $\frac{6}{7}$ (C) $\frac{6}{5}$ (D) $\frac{4}{3}$ (E) NOTA

29. The dimension of a fractal is taken to be equal to $-\frac{\ln N}{\ln \epsilon}$, where N is the number of pieces each side of one iteration is broken into to form the next iteration and ϵ is the ratio of the lengths of a side of one iteration of a fractal to the length of a side in the previous iteration. Which of the following is closest to the dimension of the Koch snowflake?

(A) $\frac{2485}{2197}$ or $\frac{540}{477} \approx 1.13$

(B) $\frac{1609}{1386}$ or $\frac{699}{602} \approx 1.16$

(C) $\frac{1386}{1099}$ or $\frac{602}{477} \approx 1.26$

(D) $\frac{1099}{693}$ or $\frac{477}{301} \approx 1.59$

(E) NOTA

30. Congratulations on making it to the end of the test! I'm sure your journey through these questions have left you rather exhausted, so to end this mathematical escapade, you simply have to evaluate the following integral! (Hint: begin with integration by parts.)

$$\int_{-1}^{0} \frac{\ln\left(x^2+1\right)}{(x+2)^2}\,dx$$

(A) $\frac{\pi - \ln 2}{20}$

(B) $\frac{\pi - 2\ln 2}{10}$

(C) $\frac{\pi - 4\ln 2}{2}$

(D) $10\ln 2 - 2\pi$

(E) NOTA