1. **(B)** Clearly the unique solution is $x = -2$.

2. **(C)** This equation factors as $(x-3)(x+1) = 0$, so the solutions are $x = -1, 3$.

3. **(B)** This equation can be written as $(x+1)^3 + (x+2)^3 = 0$, so it factors as $(2x+3)(x^2+3x+3) = 0$. The discriminant of the quadratic factor is negative, so there is one real root.

4. **(A)** This equation is equivalent to $\sin(5\theta) = \sin(\pi/2 - 3\theta)$, which will be true when $5\theta - (\pi/2 - 3\theta)$ is a multiple of 2π, which happens when θ is of the form $\frac{(4k+1)\pi}{16}$ for an integer k. Of the answers, only $\frac{2021\pi}{16}$ is of this form.

5. **(A)** Comparing exponents, what we need is that $x + 2^x = 11$, which by inspection has the solution $x = 3$.

6. **(E)** Bringing all the terms to one side, this becomes $\frac{(x-3)}{(x-0.5)(x+2)} < 0$, which has solution $(-\infty, -2) \cup (0.5, 3)$ and there are infinitely many integers in this range.

7. **(C)** We consider cases based on the sign of y. If $y \geq 0$, then $\sin x \leq y$ as long as $x \leq \arcsin y$ or $x \geq \pi - \arcsin y$. If $y < 0$, then $\sin x \leq y$ as long as $\pi + \arcsin y \leq x \leq 2\pi - \arcsin y$. In the first case, the length of the intervals is $\pi + 2\arcsin y$ and in the second case it is $\pi - 2\arcsin y$. Since $\arcsin(y) = -\arcsin(-y)$, these two are actually the same, and so the probability in either case is $\frac{\pi - 2\arcsin y}{2\pi}$.

8. **(A)** A number x is not in the domain of this function if $\sin(0.2020x) < 0$, which will first occur for positive x when $0.2020x > \pi$. This requires $2020x > 10{,}000\pi$, or $x > \frac{1000}{202}\pi \approx 5\pi$. Using the heuristic $\pi \approx 3$, we get an answer of 16, and this is correct.

9. **(C)** Solving the system by substitution gives $(x, y) = (2, -1)$.

10. **(D)** The distance from a point to a circle is the distance from a point to the center of the circle, minus the radius of the circle. Thus, all points equidistant from a line and a circle are also equidistant from the center of the circle and a copy of the line shifted perpendicularly away from the circle. The locus of points equidistant from a point and a line is a parabola, so this set is a parabola.

11. **(C)** Summing the terms in pairs gives

$$\frac{2x+5}{x^2+5x+5} + \frac{2x+5}{x^2+5x+6} = 0,$$

so $x = -5/2$ is a solution. Dividing by $2x+5$ and substituting $y = x^2 + 5x$ gives

$$\frac{1}{y+4} + \frac{1}{y+6} = 0 \implies \frac{2y+10}{(y+4)(y+6)} = 0.$$

The equation $2y + 10 = 0$ is equivalent to $x^2 + 5x + 5 = 0$, which has another two real solutions, for a total of three.

12. **(B)** This equation rearranges to $(x-2)^2 + 2\log(x-2) = 1$. Letting $y = x - 2$, this is $y^2 + 2\log y = 1$. The function $y \mapsto y^2 + 2\log y$ is strictly increasing, so this equation can have at most one real solution, and there is indeed a solution at $y = 1$ or $x = 3$.

13. **(D)** Consider matching two groups of 5 objects, say $\{1, 2, 3, 4, 5\}$ and $\{A, B, C, D, E\}$ to each other. We interpret x_{ij} as an indicator of whether object i in the first group is matched with object j in the second group. The first two equations say that each object in the first group is matched to exactly one object in the second group, and that each object in the second groups is matched to exactly one object in the third group. From here on out, we use juxtaposition to indicate a match i.e. $A1$ means matching object A with object 1.

The third equation tells us that none of $A1, B2, C3, D4, E5$ are matches while the fifth equation tells us that two of $C3, D5, E4$ are matches. This means that $D5$ and $E4$ must be matches. It remains to find the matches for A, B, and C. The fourth equation tells us that one of $B1, C4, D2, E3$ are matches. Since $D5$ and $E4$ are matches, $D2$ and $E3$ cannot be, so one of $B1$ and $C4$ is a match. But $C4$ cannot be a match, since $E4$ is a match, so the match must be $B1$.

The only possibilities left for the matches of A and C are $A2, C3$ or $A3, C2$. But we know from the third equation that $C3$ is not a match, so the final set of matches is $A3, B1, C2, D5, E4$.

With this solution, $x_{11} + x_{23} + x_{32} + x_{45} + x_{54}$ is the number of correct matches in the matching $A1, B3, C2, D5, E4$, which is 3.

14. **(B)** Let $x = 0.5 + z$ and $y = 0.5 - z$ based on the first equation. Then the second equation becomes $(2z)^{z^2+0.75} = 3^{3/4}$. Then, we substitute $z = 3^t/2$, so this becomes $3^{t(3^{2t}+3)/4} = 3^{3/4}$, so we need to take t such that $t(3^{2t} + 3) = 3$. By inspection, $t = 1/2$ works, so the solutions are $x = \frac{1}{2} + \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2} - \frac{\sqrt{3}}{2}$, which have product $-\frac{1}{2}$.

15. **(A)** The function $f(x, y)$ is equivalent to x^y. Since $(x^y)^z = x^{yz}$ but $x^{y^z} \neq (xy)^z$, only I is true.

16. **(A)** Let S be this sum. In the binary representation of S, a digit will be 1 only if, among the binary representations of the numbers 0 to 1023, that digit is 1 an odd number of times. But by symmetry, every digit in the binary representations of the numbers 1 to 1023 is 1 and 0 an equal number of times, so each appears 512 times. Thus, all the binary digits of S are 0, and so $S = 0$.

17. **(D)** Note that

$$\sin\theta(4\cos^2\theta - 1) = 2\sin\theta\cos^2\theta + 2\sin\theta\cos^2\theta - \sin\theta = \sin(2\theta)\cos\theta + \sin\theta\cos(2\theta) = \sin(3\theta),$$

which is the equation of a polar rose.

18. **(C)** The quantity $x^T Ax$ is equal to $\sum_{i,j} x_i x_j a_{i,j}$ where $x_i, x_j, a_{i,j}$ are the entries of x and A respectively. In our case, this sum has 100 terms, each of which is equal to $2 \cdot 2 \cdot 3 = 12$, so the sum is 1200 and the answer is 3.

19. **(B)** Since both terms on the LHS must be positive integers, they are either 2 and 0 or 1 and 1 in some order. If they are 1 and 1, this means that $x = y$, giving 99 solutions. If they are 2 and 0, this means that $y^2 \leq x \leq y^3 - 1$ or $x^2 \leq y \leq x^3 - 1$. There are 321 such pairs, and after we account for ordering x, y there are 642. This gives a total of 741 solutions, and an answer of 12.

20. **(C)** Linear functions are strictly increasing or decreasing, so the maximum value of $2x - y$ must occur at a corner of the region defined by these inequalities. The corners of this region are at the points $(0, -1), (0, 5/3), (1, -1), (25/11, 10/11)$. By checking the value of $2x - y$ at each point, we find that the maximum occurs at the final point, so the maximum value is $\frac{40}{11}$ and the answer is 51.

21. **(E)** None of these is true. First, consider the equations $x + y = 1, 2x + 2y = 2$. Then $m = n$, but there is no unique solution. Second, consider $x + 1 = 2$ and $2x + 2 = 4$. Then $m < n$, but there are solutions. Finally, consider $x + y + z = 1, 2x + 2y + 2z = 3$. Then $m > n$, but there are no solutions.

22. **(B)** Note that

$$\frac{\sin(3\theta)}{\sin(\theta)} = \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} = 3 - 4\sin^2\theta,$$

and then

$$3 - 4\sin^2\theta = \frac{1}{5} \implies \sin^2\theta = \frac{7}{10}.$$

Thus

$$\frac{\cos(3\theta)}{\cos\theta} = \frac{4\cos^3\theta - 3\cos\theta}{\cos\theta} = 4\cos^2\theta - 3 = -\frac{9}{5},$$

and the answer is 14.

23. **(E)** Let $z = re^{-i\theta}$ for some $r > 0$. Then $z^{|z|} = r^r e^{ir\theta}$. Clearly we can obtain any value of $r\theta$, but r^r will always be positive, so the range is the entire plane except the origin.

24. **(B)** Examining the graphs gives 2 points of intersection.

25. **(A)** Persistent manipulation of the given expression is necessary. For convenience, let $y = \sqrt{1+\tan x} + \sqrt{1+\sin x}$. Note that as x goes to 0, y goes to 2. Then:

$$\frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} \cdot \frac{y}{y} = \frac{\tan x - \sin x}{x^3 y} = \frac{\sin x(\sec x - 1)}{x^3 y} = \frac{\sin x}{x} \cdot \frac{\sec x - 1}{x^2 y}.$$

As x goes to 0, $\dfrac{\sin x}{x}$ goes to 1, so we may ignore it as we proceed:

$$\frac{\sec x - 1}{x^2 y} \cdot \frac{\cos x}{\cos x} = \frac{1-\cos x}{x^2 y \cos x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{\sin^2 x}{x^2} \cdot \frac{1}{y \cos x(1+\cos x)}.$$

As x goes to 0, the first term again goes to 1, and direct substitution shows the second goes to $\dfrac{1}{4}$.

26. **(B)** For a given root, say r, $r^3 = 11r + 5$, so $r^5 = 11r^3 + 5r^2 = 11(11r+5) + 5r^2 = 5r^2 + 121r + 55$. The sum of the roots is 0 and the sum of the squares of the roots is 22, so the answer is $5 \cdot 22 + 3 \cdot 55 = 275$.

27. **(D)** First, let $\theta = \frac{\pi}{4} + \arctan \frac{1}{239}$, so we want $\tan\left(\frac{\theta}{4}\right)$. Then,

$$\tan\theta = \frac{1 + \frac{1}{239}}{1 - \frac{1}{239}} = \frac{120}{119}.$$

Based on the given triple, this means that $\sin\theta = \frac{120}{169}$ and $\cos\theta = \frac{119}{169}$. Next, it is well known that

$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} \implies \tan\frac{x}{4} = \frac{1-\cos\frac{x}{2}}{\sin\frac{x}{2}} = \frac{1-\sqrt{\frac{1+\cos x}{2}}}{\sqrt{\frac{1-\cos x}{2}}}.$$

Now we just plug in our values and the value is

$$\frac{1-\sqrt{\frac{1+\frac{119}{169}}{2}}}{\sqrt{\frac{1-\frac{119}{169}}{2}}} = \frac{1-\sqrt{\frac{144}{169}}}{\sqrt{\frac{25}{169}}} = \frac{\frac{1}{13}}{\frac{5}{13}} = \frac{1}{5},$$

so the answer is 6.

28. **(D)** Suppose n has 4 digits - then the maximum digit sum is 36, but $19 \cdot 36 < 10^3$ so there are no solutions. Now we can let $n = 100a + 10b + c$ for (possibly zero) digits a, b, and c. Then, we want

$$100a + 10b + c = 19a + 19b + 19c \implies 81a = 9b + 18c \implies 9a = b + 2c.$$

Since b and c are digits, the maximum value of $b + 2c$ is 27. This restricts us to $a = 0, 1, 2, 3$. Checking each of these cases, we find 11 solutions.

29. **(B)** We have

$$\lfloor x^2 \rfloor - \lfloor x \rfloor^2 = x^2 - \{x^2\} - \lfloor x \rfloor^2 = (\lfloor x \rfloor + \{x\})^2 - \{x^2\} - \lfloor x \rfloor^2 = 2\lfloor x \rfloor\{x\} + \{x\}^2 - \{x^2\},$$

so

$$\frac{\lfloor x^2 \rfloor - \lfloor x \rfloor^2}{\lfloor x \rfloor\{x\}} = 2 + \frac{\{x\}^2 - \{x^2\}}{\lfloor x \rfloor\{x\}}.$$

The second term is clearly always less than 1, so the largest the ceiling of this expression can be is 3.

30. **(B)** First, note that if $\tau(n) = 4$, then either $n = p^3$ or $n = pq$ for primes p, q. Then, if $\tau(n) = p^3$, either $n = a^{p^3-1}$ or $n = a^{p^2-1}b^{p-1}$ for primes a, b and if $\tau(n) = pq$ then either $n = a^{pq-1}$ or $n = a^{p-1}b^{q-1}$. Checking all these cases with $p = 2, q = 3$ and $a = 2, b = 3$, which will give the smallest values, we find an answer of 12.