Answer Key: ADABC-DDBAD-DBBEB-BDCCB-CACDE-BDDCA

Solutions:

1.) A,

Note: $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

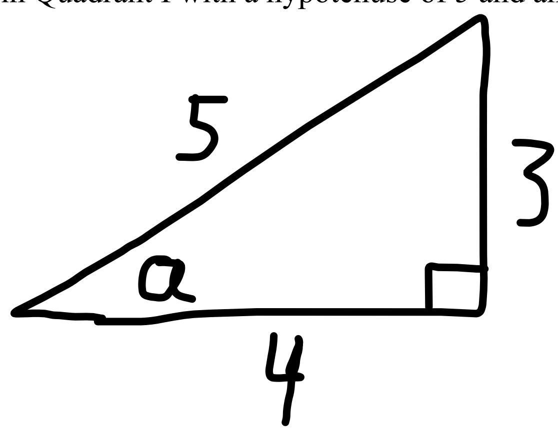
$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right)$$

$$\sin\left(\frac{7\pi}{12}\right) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

2.) D,

$$\frac{\csc x}{\cot x} \cdot \frac{\sin x}{\tan x} = \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}} \cdot \frac{\frac{\sin x}{1}}{\frac{\sin x}{\cos x}} = \frac{1}{\cos} x \cdot \cos x = 1$$

3.) A, by constructing a right triangle in Quadrant I with a hypotenuse of 5 and an opposite side of 3 (using angle a) We find that:



$$\cot a = \frac{\cos a}{\sin a} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$$

$$\csc a = \frac{1}{\sin a} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

$$\tan a = \frac{\sin a}{\cos a} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$\sec a = \frac{1}{\cos a} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

Thus, $(\cot\alpha + \csc\alpha) - (\tan\alpha + \sec\alpha) = \left(\frac{4}{3}+\frac{5}{3}\right) - \left(\frac{3}{4}+\frac{5}{4}\right) = \frac{9}{3} - \frac{8}{4} = 3 - 2 = 1$

4.) B,

Using the property that $\sin^2 x + \cos^2 x = 1$ we find that $\sin^2 x = 1 - \cos^2 x$. Using substitution:

$$1 - \cos^2 x = \cos^2 x$$

$$2\cos^2 x = 1$$

$$\cos^2 x = \frac{1}{2}$$

$$\cos x = \pm\sqrt{\frac{1}{2}}$$

$$\cos x = \pm\frac{\sqrt{2}}{2}$$

This occurs at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ within our domain. The sum of which is $\frac{(1+3+5+7)\pi}{4} = \frac{16\pi}{4} = 4\pi$. If we were to check each solution, we would find that all solutions work and there are no extraneous solutions.

5.) C, since we know that 3 = $\sqrt{9}$ and 5 = $\sqrt{25}$ we realize that the angle opposite the side of $\sqrt{19}$ is the second largest angle. Thus, we can apply the law of cosines to find it.

$c^2 = a^2 + b^2 - (2)(a)(b)(\cos \mathrm{C})$

$\left(\sqrt{19}\right)^2 = 3^2 + 5^2 - (2)(3)(5)(cos\ \mathrm{C})$

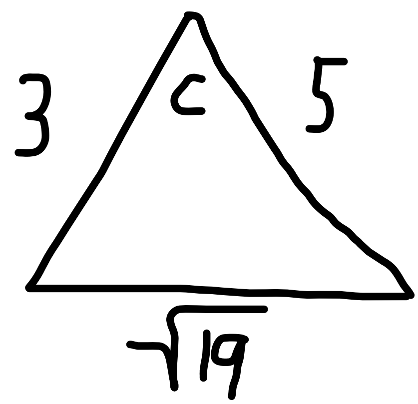
$19 = 9 + 25 - 30 \cos \mathrm{C}$

$-15 = -30 \cos \mathrm{C}$

$\frac{-15}{-30} = \cos \mathrm{C}$

$\cos \mathrm{C} = \frac{1}{2}$

$\mathrm{C} = \frac{\pi}{3}$.



Thus, a = 1 and b = 3. 1(5) + 3 = 8

6.) D, by definition $\cos 2x = \sin^2 x - \cos^2 x$ and by manipulating this equation using $\sin^2 x + \cos^2 x = 1$ we find that the other forms are $2\cos^2\theta - 1$ and $1 - 2\sin^2\theta$. $2 \cos x \sin x$ is the formula for $\sin 2x$.

7.) D, the first thing to do is to figure out the probabilities of Mr. Otto's die. Since it is kth times more likely to roll a k than 1, then it is 6 times more likely to roll a 6 than a 1 for example. This holds true for number 1-6 meaning that we will need a total of (6+5+4+3+2+1) slots to visualize the probability which is equal to 21. So, the probabilities are as follows:

$P(1) = \frac{1}{21}$

$P(2) = \frac{2}{21}$

$P(3) = \frac{3}{21}$

$P(4) = \frac{4}{21}$

$P(5) = \frac{5}{21}$

$P(6) = \frac{6}{21}$

So, now we can calculate the probability of Mr. Otto winning we will be doing it piece by piece (all the scenarios where Mr. Snow rolls from 1-6). Since Mr. Snow's probabilities are all the same each one has a 1/6 chance of being rolled, we then multiply this by the probability that Mr. Otto's die is 2 or more. We then sum all of these probabilities to find the probability of Mr. Snow. Let X represent Mr. Otto's die and let Y represent Mr. Snow's die.

$P(X \geq 3 \; and \; Y = 1) \; = \; \left(\frac{1}{6}\right)\left(\frac{3}{21} + \frac{4}{21} + \frac{5}{21} + \frac{6}{21}\right) = \frac{18}{126}$

$P(X \geq 4 \; and \; Y = 2) \; = \; \left(\frac{1}{6}\right)\left(\frac{4}{21} + \frac{5}{21} + \frac{6}{21}\right) = \frac{15}{126}$

$P(X \geq 5 \; and \; Y = 3) \; = \; \left(\frac{1}{6}\right)\left(\frac{5}{21} + \frac{6}{21}\right) = \frac{11}{126}$

$PX \geq 6 \; and \; Y = 4) \; = \; \left(\frac{1}{6}\right)\left(\frac{6}{21}\right) = \frac{6}{126}$

$P(X \geq 7 \; and \; Y = 5) = \; \left(\frac{1}{6}\right)(0) = 0$

$P(X \geq 8 \; and \; Y = 6) = \; \left(\frac{1}{6}\right)(0) = 0$

$P(X \geq Y + 2) = \; \frac{18 + 15 + 11 + 6}{126} = \frac{25}{63}$

25 + 63 = 88

8 + 8 = 16

8.) B,

$\tan^2\theta = 0$

$\tan\theta = \; \pm\sqrt{0}$

$\tan\theta = 0$

This occurs at 0 and π and therefore there are 2 solutions within our domain.

9.) A, Using the property that $\sin^2 x + \cos^2 x = 1$ we find that $\cos^2 x \; = 1 - \; \sin^2 x$

$\cos^2 x \; - 3\sin x \; = \; -3$

$1 - \; \sin^2 x - 3\sin x = \; -3$

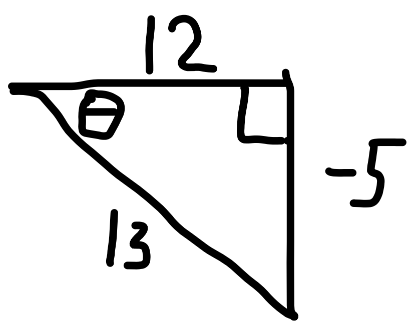
$\sin^2 x + 3\sin x - 4 = 0$

$(\sin x + 4)(\sin x - 1) = 0$

$\sin x = 1$ using only the $(\sin x - 1)$ as it lies within our domain

Within our domain this occurs at $\frac{\pi}{2}$.

10.) D, firstly we need to know that $\sin 2\theta \; = 2\cos\theta \sin\theta$. By constructing a right triangle in the fourth quadrant we find that:



$\sin\theta = \frac{-5}{13}$

Thus,

$\sin 2\theta = (2)\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) = \frac{-120}{169}$

$a = \; -120$ and $b = 169$

$a + b = 169 - 120 = 49$. The units digit is 9.

11.) D,

$2\sin^2 x \; - \sin x \; - \; 1 \; = \; 0$

$(\sin x - 1)(2\sin x + 1) = 0$

$\sin x = 1$ This occurs at $\frac{\pi}{2}$ within our domain.

$\sin x = \frac{-1}{2}$ This occurs at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$ within our domain.
The sum of $\frac{\pi}{2} + \frac{7\pi}{6} + \frac{11\pi}{6} = \frac{7\pi}{2}$. 7+2 = 9.

12.) B,

Using the law of sines:

$$\frac{\sin a}{a} = \frac{\sin b}{b}$$

$$\frac{\sin 120^o}{b} = \frac{\sin 30^o}{2}$$

$$\frac{\frac{\sqrt{3}}{2}}{b} = \frac{\frac{1}{2}}{2}$$

$$\sqrt{3} = \frac{b}{2}$$

$$b = 2\sqrt{3}$$

13.) B

To find the area of a triangle given its side lengths, we use Heron's Theorem. Thus,

$$s = \frac{2+4+4}{2} = \frac{10}{2} = 5$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$A = \sqrt{5(5-4)(5-4)(5-2)}$$

$$A = \sqrt{5(1)(1)(3)}$$

$$A = \sqrt{15}$$

14.) E, by plotting these points we find that they are co-linear meaning that no triangle can be constructed which constitutes to an area of 0. However, if this wasn't obvious, we can find it the long way.

$$A = \frac{1}{2}|\det|$$

det= $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 1 \\ 1 & 4 & 1 \end{bmatrix}$ = (10+3+0)-(5+8+0) = 0

$$A = \frac{1}{2}|0| = \frac{1}{2}(0) = 0$$

15.) B, using a systematic thinking style, assume that each letter is true until it is proven that it isn't. The main take away is that only ONE answer can be correct.

If A is true, then B is falsely states that C is false making C also correct which makes D correct which makes E correct, but this is problematic as the answer A,C, D, and E are all "solutions" if A is assumed to be true, however we cannot have all 4 correct answers and E contradicts A,C and D. Therefore, A cannot be true.

If B is true, then C must be false which makes D false as well which makes E false as well meaning that there is potentially one “solution.” By nature, if B is true then A must be false. Answer B works and is our answer.
If C is true, then D would also be true which makes E true as well, similar to the problem if we assumed that A was true, we get the potential solutions of C,D and E but E contradicts both C and D. Therefore, C cannot be true.
If D is true, then E is also true. Once again, we run into the problem of E contradicting D. Therefore, D cannot be true.
If E is true, then letters A-D must be say the wrong statement. So, A now states that B is true, but if B is true according to A, but according to E it’s statement is false. In this case, E makes A false but by doing so A makes B true which goes against the answer of “NOTA.” Therefore, E cannot be true.
Thus, B is the only answer choice.

16.) B,

$$\frac{1}{10}+\frac{1}{5}+\frac{1}{5}=\frac{1}{h}$$
$$\frac{1}{10}+\frac{2}{10}+\frac{2}{10}=\frac{1}{h}$$
$$\frac{1}{2}=\frac{1}{h}$$
$$h=2.$$

17.) D,

first, we convert both the polar coordinates to Cartesian coordinates, then we use the distance formula. The respective formulas that we will be using are:

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ (distance formula)

$x=r\cdot cos\,\theta$ (polar to Cartesian formula for x coordinate)

$y=r\cdot sin\,\theta$ (polar to Cartesian formula for y coordinate)

Thus, $\left(5,\frac{\pi}{3}\right)=(r,\theta)$.

$$x=5\cos\frac{\pi}{3}$$
$$x=5\left(\frac{1}{2}\right)$$
$$x=\frac{5}{2}$$
$$y=5\sin\frac{\pi}{3}$$
$$y=\frac{5\sqrt{3}}{2}$$

Thus, the coordinate is $\left(\frac{5}{2},\frac{5\sqrt{3}}{2}\right)$

For our second coordinate we use the same process. Thus, $\left(2,-\frac{\pi}{4}\right)=(r,\theta)$. Using co-terminal angles we find that we can add 2π to our $-\frac{\pi}{4}$ to make our calculations easier, getting a value of $\frac{7\pi}{4}$.

$x=2\cos\frac{7\pi}{4}$

$x=2\left(\frac{\sqrt{2}}{2}\right)$

$x=\sqrt{2}$

$y=2\sin\frac{7\pi}{4}$

$y=-\sqrt{2}$

Thus, the coordinate is $(\sqrt{2},-\sqrt{2})$

Now using the distance formula:

$$distance=\sqrt{\left(\frac{5}{2}-\sqrt{2}\right)^2+\left(\frac{5\sqrt{3}}{2}--\sqrt{2}\right)^2}$$

But since we are finding the square of the distance or d^2we square both sides getting:

$$distance^2=\left(\frac{5}{2}-\sqrt{2}\right)^2+\left(\frac{5\sqrt{3}}{2}--\sqrt{2}\right)^2$$

$$distance^2=\left(\frac{5}{2}-\sqrt{2}\right)^2+\left(\frac{5\sqrt{3}}{2}+\sqrt{2}\right)^2$$

Now simplify using $(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$

$$distance^2=\frac{25}{4}-(2)\frac{5\sqrt{2}}{2}+2+\frac{75}{2}+(2)\frac{5\sqrt{6}}{2}+2$$

$$distance^2=\frac{25}{4}-5\sqrt{2}+2+\frac{75}{2}+5\sqrt{6}+2$$

$$distance^2=\frac{191}{4}-5\sqrt{2}+0\sqrt{3}+5\sqrt{6}$$

Now that we have find $distance^2$ we find $4a+3b+2c+d$. Thus, $4a+3b+2c+d=4\left(\frac{191}{4}\right)+3(-5)+2(0)+5=191-15+5=181$. The sum of the digits is equal to 1+8+1 = 10.

18.) C,

To evaluate this sum we need to know how i^{nth} powers interact. Writing down the first few powers of i and noting their values we find that $i^1=i, i^2=-1, i^3=-i, i^4=1, i^5=i^4\cdot i=1i=1$. This pattern continues and we see that the cycle repeats every 4th power. Thus, we need to find the remainder of the power when it is divided by 4. Luckily for us the division rule for 4 is simply the last 2 digits. Thus,

$(i^{1010} \cdot\ i^{2020}\ \cdot i^{3030}\ \cdot i^{4040}) - (i^{1010} +\ i^{2020}\ + i^{3030}\ + i^{4040})$

$(i^2 \cdot i^4 \cdot i^2 \cdot i^4) - (i^2 + i^4 + i^2 + i^4)$

$(1) - (-1 + 1 - 1 + 1) = 1$

19.) C, the function is a circle whose center is at the origin and has a radius of 1 (it's the unit circle). This circle is considered to be both odd and even as it is symmetric about both the origin and y-axis respectively. Furthermore, it satisfies both the odd and even rule for all real x.

Odd rule: -f(x) = f(-x)

If a function is odd, then it is symmetric about the line y=x. The unit circle clearly is.

Even rule: f(x) = f(-x)

If a function is even, then it is symmetric about the y-axis. The unit circle clearly is.

20.) B, using the mnemonic CEPH we can easily determine the eccentricity any regular conic section. Let e be the eccentricity of a conic section. Thus,

Circle: e = 0

Ellipse: 0<e<1

Parabola: e = 1

Hyperbola: e>1

Therefore, Nahim is correct.

21.) C, when written in the form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ the value of the determinant is ad-bc. Using this, we find that $\begin{vmatrix} 7 & e \\ \pi & 4 \end{vmatrix} = (7)(4) - \pi e = 28 - \pi e$ and $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = (1)(1) - (0)(-1) = 1$. Thus, we have the product of $(28 - \pi e)(1)^{999} = 28 - \pi e$. Thus, $\alpha = 28$ and $\beta = -1$. The sum of which is 27.

22.) A,

$\log_2(x + 2) + \log_2(5) = 4$

$\log_2(5x + 10) = 4$

$5x + 10 = 16$

$5x = 6$

$x = \frac{6}{5}$

23.) C,

$\frac{6x^{-2} + 9x^2}{3x^{-2}} = \frac{6x^{-2}}{3x^{-2}} + \frac{9x^2}{3x^{-2}} = 2 + 3x^4$

24.) D,

$2^{2k - 1} = 8^{k + 5}$

$2^{2k - 1} = 2^{3(k + 5)}$

$2k - 1 = 3k + 15$

$k = -16$

25.) E, to find the tens digit of the cube of the distance we need to first put both conics in their standard forms. Thus,

$2x^2 + 2y^2 - 8x + 4y + 2 = 0$

$2(x^2 - 4x + 4) + 2(y^2 + 2y + 1) = -2 + (2)(4) + 2(1)$

$2(x-2)^2 + 2(y+1)^2 = 8$

$(x-2)^2 + (y+1)^2 = 4$

The center is (2,-1).

$x^2 - 4y^2 + 4x + 16y - 21 = 0$

$(x^2 + 4x + 4) - 4(y^2 - 4y + 4) = 21 + (1)(4) + (-4)(4)$

$(x+2)^2 - 4(y-2)^2 = 9$

The center is (-2,2).

Using the distance formula between two points we have $\sqrt{(2 - -2)^2 + (-1-2)^2} = \sqrt{16+9} = \sqrt{25} = 5.$

Thus, A = 5. The cube of A = (5)(5)(5) = 125. The tens digit is 2.

26.) B, because the function is a parabola we know that the x vertex of a parabola that is written in terms of x we know that its value is the midpoint between the two roots. In our case the roots are m and n. The midpoint of these values is $\frac{m+n}{2}$.

Alternative Solution:

By foiling the function, we find that f(x) = $-2x^2 + 2x(m+n) - 2mn$ and using $\frac{-b}{2a} = \frac{-2(m+n)}{2(-2)} = \frac{m+n}{2}$.

27.) D, making a 2X2 determinant with variables $\begin{vmatrix} x & y \\ z & w \end{vmatrix}$ will let us prove the value of the inverse of the transpose of T.

Let A = $\begin{vmatrix} x & y \\ z & w \end{vmatrix}$.

The inverse of A (denoted as A^{-1}) = $\frac{1}{xw-yz}\begin{vmatrix} w & -y \\ -z & x \end{vmatrix}$.

The transpose of the inverse of A (denoted as $(A^{-1})^T = \frac{1}{xw-yz}\begin{vmatrix} w & -z \\ -y & x \end{vmatrix}$. Solving this we get: $\frac{1}{xw-yz}(wx - yz) = 1$. Thus, the determinant of the transpose of the inverse of $A = \frac{1}{A} = A$.

Since $\frac{1}{A} = T$, then T turns into $\frac{1}{T}$.

28.) A, we simply have to do both operations giving us two equations then we solve for the 2 unknowns. Thus,

$\vec{a} \cdot \vec{b} = (1)(2) - x + y$

$6 = 2 - x + y$

$4 + x = y$ (we will use this substitution later)

$$|\vec{a} \times \vec{b}| = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & x & y \end{vmatrix}$$

$|\vec{a} \times \vec{b}| = i(-y-x) + j(-y+2) + k(x+2)$
$12 = \sqrt{(-y-x)^2 + (-y+2)^2 + (x+2)^2}$
$144 = x^2 + 2xy + y^2 + y^2 - 4y + 4 + x^2 + 4x + 4$
$144 = 2x^2 + 2xy + 2y^2 + 4x - 4y + 8$
$144 = 2x^2 + 2x(4+x) + 2(x+4)^2 + 4x - 4(x+4) + 8$
$2x^2 + 8x + 2x^2 + 2(x^2 + 8x + 16) + 4x - 4x - 16 + 8 - 144 = 0$
$6x^2 + 24x - 120 = 0$
$x^2 + 4x - 20 = 0$
$x = \frac{-4 \pm \sqrt{16 - 4(1)(-20)}}{2(1)}$
$x = \frac{-4 \pm 4\sqrt{6}}{2}$
$x = -2 \pm 2\sqrt{6}$
$y = -2 \pm 2\sqrt{6} + 4$ (since we know that y = x+4)
$y = 2 \pm 2\sqrt{6}$
$\alpha + \beta + \delta + \varepsilon = 2 + 6 - 2 + 2 = 8$
29.) C, in order to solve absolute value equations we must test all the cases (+-, ++, --, -+). To test a positive case we leave the expression as is and to test a negative expression we distribute a negative one to each term. Make sure to recheck answers as they may be extraneous.
$|2x + 1| + |x - 3| = 9$

$2x + 1 + x - 3 = 9$
$3x = 11$
$x = \frac{11}{3}$ (++ case) (this solution exists as it does satisfy the original equation)
$2x + 1 - x + 3 = 9$
$x = 5$ (+- case) (this solution is extraneous as it does not satisfy the original equation)
$-2x - 1 + x - 3 = 9$
$-x = 13$
$x = -13$ (-+ case) (this solution is extraneous as it does not satisfy the original equation)
$-2x - 1 - x + 3 = 9$
$-3x = 7$
$x = \frac{-7}{3}$ (-- case) (this solution exists as it does satisfy the original equation)
Thus, there are 2 real solutions.
30.) A,
I is false as not all parabolas have two roots. The parabola $y = x^2$ only has 1 root at $x = 0$.
II is false as we can not conclude this of all function f(x). For example, a parabola would not fit this statement.
III (a+bi)(a-bi) = $a^2 + b^2$ which is always real.