Calculus – February Regional – Individual Test – 2021

Answers:

1. C	6. B	11. D	16. D	21. B	26. B
2. D	7. C	12. B	17. B	22. C	27. A
3. E	8. B	13. C	18. A	23. C	28. D
4. A	9. A	14. C	19. B	24. B	29. E
5. D	10. B	15. A	20. B	25. D	30. C

Solutions:

1. **C** Evaluate the limit: $\lim\limits_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-4}-\sqrt{x-4}}{(\Delta x+x)-(x)}$

This is the definition of a derivative:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

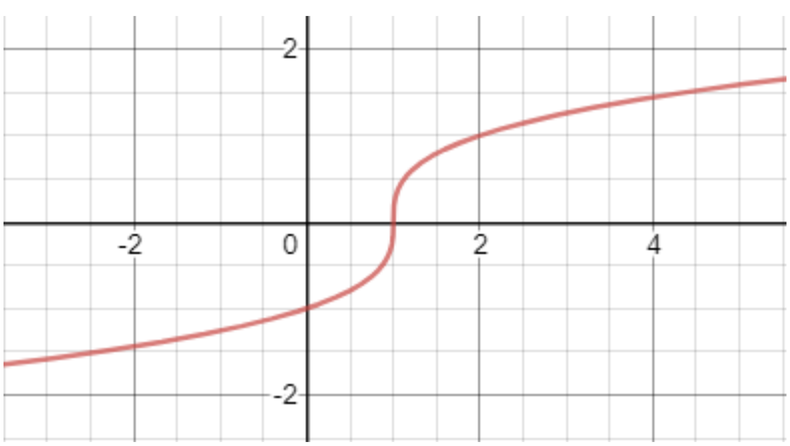
$$f(x) = (x-4)^{\frac{1}{2}} \qquad f'(x) = \frac{1}{2}(x-4)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-4}} = \frac{\sqrt{\boldsymbol{x-4}}}{\boldsymbol{2(x-4)}}$$

2. **D** Find the values of a such that: $\lim\limits_{x \to a} \frac{4+x^2}{\sqrt{4-x^2}} = \infty$ or $\lim\limits_{x \to a} \frac{4+x^2}{\sqrt{4-x^2}} = -\infty$.

This is the definitional of vertical asymptotes, which would be located at any place where the denominator of the rational function is 0. Those are **x = -2 and 2.**

3. **E** Determine if the function is differentiable. If not, give the reason why.

$$y = \sqrt[3]{x-1} \ when\ x = 1$$



Yes, the graph is continuous and smooth. But, the algebraic derivative at x=1 is undefined. The two different one-sided limits as x approaches 1 of the derivative do not exist. So, the function is not differentiable at x=1. So, "yes" cannot be the correct answer -- the function is not differentiable there. The three "no" answers do not correctly describe the reason that the function is differentiable.

4. **A** Use the table of values for a continuous function f(x) to find the average rate of change on the interval [2,6].

x	0	2	4	6	8	10
$f(x)$	32	24	12	-4	-20	-36

Average rate of change is the slope of the line between the two points: (2,24) to (6, -4) = $\frac{-4-24}{6-2} = -\frac{28}{4} = -7$

5. **D** The base of a right cylindrical tank has a perimeter of 48π feet. The height of the cylinder is 10 yards. The tank is filled to a depth of 4 feet and is flowing into the tank at a rate of 2 ft^3/sec. Find the rate of change of the depth of the water in the tank.

$$V = \pi r^2 h \;;\quad \frac{dV}{dt} = 2\pi r \frac{dr}{dt}(h) + \pi r^2 \frac{dh}{dt} \;; the\ radius\ does\ not\ change, so \frac{dr}{dt} = 0$$

$$48\pi = 2\pi r\,; r = 24 \qquad 2 = \pi(24)^2 \frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{1}{288\pi}$$

6. **B** The Atlanta Braves ticket salesmen have found that if they sell tickets for $40 each, they can sell 12,500 tickets, but for each $5 they raise the price, 500 less people attend. What price should the Atlanta Braves sell the tickets at to maximize revenue (amount of money brought in)?

Attendance = 12,500 – 500x

Price = 40 + 5x

Revenue = (12,500 – 500x)(40 + 5x) = (500)(25-x)(5)(8+x)

Revenue = 2500 [200+17x-x^2]

R' = 2500 [17-2x]

To find the maximum, set the derivative equal to zero, 0 = 17-2x ; x = 8.5

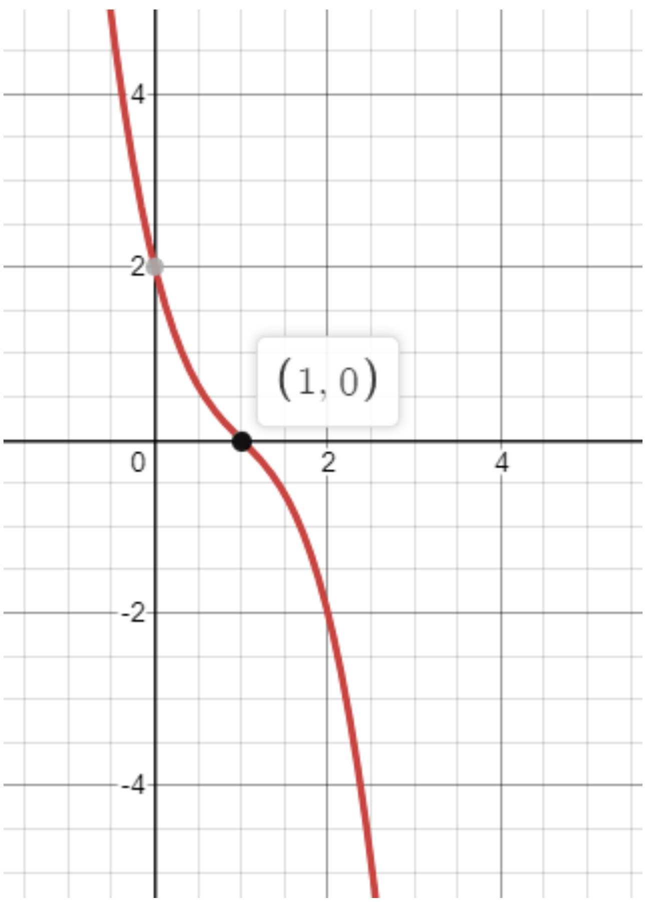
Substitute that into the price equation to get: 40 +(5)(8.5) = **$82.50**

7. **C** Compute the derivative of $y = \frac{x^3}{\ln(x)}$ at $x = e^2$.

$$y = \frac{x^3}{\ln(x)}\ ;Quotient\ Rule \quad y' = \frac{3x^2 lnx - \frac{1}{x}(x^3)}{(lnx)^2}$$

$$y'(e^2) = \frac{3(e^2)^2\ln(e^2) - \frac{1}{(e^2)}((e^2)^3)}{(\ln(e^2))^2} = \frac{3e^4(2) - e^4}{4} = \frac{5e^4}{4}$$

8. **B** Where is the graph of the equation $y = -x^3 + 3x^2 - 4x + 2$ concave up?



To determine concavity, we need to find the second derivative to solve for possible points of inflection:

$$y = -x^3 + 3x^2 - 4x + 2\ ; y' = -3x^2 + 6x - 4\ ; y'' = -6x + 6\ ;\ 0 = -6x + 6;\ x = 1$$

By checking values on either side of the number line for the second derivative, we can see that the concavity changes from positive to negative at x = 1. Therefore the interval where it is positive is $(-\infty, 1)$

9. **A** Find the sum of all values of x that satisfy the Mean Value Theorem of Derivatives for the equation: $f(x) = \frac{x^2+1}{-x}$ on the interval [2, 3].

First we must find the slope of the secant line between x = 2 and x = 3. To find the y-values we will sub into the original equation. (2, -5/2) and (3, -10/3)

Using the slope formula:

$$\frac{-\frac{10}{3} - \left(-\frac{5}{2}\right)}{3-2} = \frac{\frac{-20+15}{6}}{1} = -\frac{5}{6}$$

Next, we will find the derivative to find the equation of the slope of the tangent line to the graph:

$$f'(x) = \frac{(2x)(-x) - (-1)(x^2+1)}{(-x)^2} = \frac{-2x^2 + x^2 + 1}{x^2} = \frac{-x^2+1}{x^2} = -\frac{5}{6}$$

$$\frac{-x^2+1}{x^2} = -\frac{5}{6}; (-6)(-x^2+1) = 5x^2; 6x^2 - 6 = 5x^2; x^2 = 6; x = \pm\sqrt{6}$$

The only value on the interval given is $\sqrt{6}$.

10. **B** Evaluate: $D_t \int_3^{4t^2-2t} 3x^2 - \frac{1}{5}x\,dx$

$$\left[3(4t^2-2t)^2 - \frac{1}{5}(4t^2-2t)\right](8t-2); \left(48t^4 - 48t^3 \mp 12t^2 - \frac{4}{5}t^2 + \frac{2}{5}t\right)(8t-2)$$

$$\left(48t^4 - 48t^3 \mp \frac{56}{5}t^2 + \frac{2}{5}t\right)(8t-2) = 384t^5 - 384t^4 + \frac{448}{5}t^3 + \frac{16}{5}t^2 - 96t^4 + 96t^3 - \frac{112}{5}t^2 - \frac{4}{5}t$$

$$384t^5 - 480t^4 + \frac{928}{5}t^3 + -\frac{96}{5}t^2 - \frac{4}{5}t$$

11. **D** Find: $\int \frac{x^4-3x^3+2x^2-x+1}{x^2-2x-1}dx$

Using long division:

$$
\begin{array}{r|rrrrr}
 & x^2 & -x & +1 & & \\
\hline
x^2-2x-1 & x^4 & -3x^3 & +2x^2 & -x & +1 \\
- & x^4 & -2x^3 & -x^2 & & \\
\hline
 & & -x^3 & +3x^2 & -x & +1 \\
- & & -x^3 & +2x^2 & +x & \\
\hline
 & & & x^2 & -2x & +1 \\
- & & & x^2 & -2x & -1 \\
\hline
 & & & & & 2
\end{array}
$$

Hints

$\frac{x^4}{x^2}=x^2$

$x^2\left(x^2-2x-1\right)=x^4-2x^3-x^2$

$\frac{-x^3}{x^2}=-x$

$-x\left(x^2-2x-1\right)=-x^3+2x^2+x$

$\frac{x^2}{x^2}=1$

$1\left(x^2-2x-1\right)=x^2-2x-1$

Therefore, $\frac{x^4-3x^3+2x^2-x+1}{x^2-2x-1}=x^2-x+1+\frac{2}{x^2-2x-1}$

Answer: $\frac{x^4-3x^3+2x^2-x+1}{x^2-2x-1}=x^2-x+1+\frac{2}{x^2-2x-1}$

12. **B** Evaluate: $\int \frac{dx}{4+9x^2}$

Problem:

$$\int \frac{1}{9x^2+4}\,\mathrm{d}x$$

Substitute $u = \frac{3x}{2} \longrightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{2}$ (steps) $\longrightarrow \mathrm{d}x = \frac{2}{3}\,\mathrm{d}u$:

$$= \int \frac{2}{3\left(4u^2+4\right)}\,\mathrm{d}u$$

Simplify:

$$= \frac{1}{6}\int \frac{1}{u^2+1}\,\mathrm{d}u$$

Now solving:

$$\int \frac{1}{u^2+1}\,\mathrm{d}u$$

This is a standard integral:

$$= \arctan(u)$$

Plug in solved integrals:

$$\frac{1}{6}\int \frac{1}{u^2+1}\,\mathrm{d}u$$

$$= \frac{\arctan(u)}{6}$$

Undo substitution $u = \frac{3x}{2}$:

$$= \frac{\arctan\left(\frac{3x}{2}\right)}{6}$$

The problem is solved:

$$\int \frac{1}{9x^2+4}\,\mathrm{d}x$$

$$= \frac{\arctan\left(\frac{3x}{2}\right)}{6} + C$$

13. **C** Find $\frac{dy}{dx}$ if $\tan(x+y) = 2x$.

$$\tan(x+y) = 2x \ ; \frac{tanx + tany}{1 - tanxtany} = 2x \ ; tanx + tany = 2x(1 - tanxtany)$$

Differentiate:

$$\sec^2 x + \sec^2 y \frac{dy}{dx} = 2(1 - tanxtany) + (-1)(2x)(\sec^2 xtany + \sec^2 y \frac{dy}{dx} tanx)$$

$$\frac{dy}{dx}[\sec^2 y + 2xtanxsec^2 y] = 2 - 2tanxtany - 2xsec^2 xtany - \sec^2 x$$

$$\frac{dy}{dx} = (2 - 2tanxtany - 2xsec^2 xtany - \sec^2 x)/(\sec^2 y + 2xtanxsec^2 y)$$

14. **C** Evaluate: $\int_{e^2}^{e^3} (\ln x)^3 dx$

$$t = lnx \quad dt = \frac{1}{x} dx \quad x = e^t \quad dx = e^t dt$$

$\int t^3 [e^t dt]$ Using By Parts Integration:

$$u = t^3 \ ; du = 3t^2 \ ; v = e^t \ ; dv = e^t dt$$

$$t^3 e^t - \int e^t 3t^2 dt$$

By Parts again: $u = 3t^2; du = 6t \ ; v = e^t; dv = e^t dt$

$$t^3 e^t - [3t^2 e^t - \int e^t 6t dt]$$

By Parts again: $u = 6t \ ; du = 6dt \ ; v = e^t \ ; dv = e^t dt$

$$t^3 e^t - 3t^2 e^t + [6te^t - \int e^t 6dt]$$

By Parts again: u = 6 ; du = 0 ; $v = e^t \ ; dv = e^t dt$

$$t^3 e^t - 3t^2 e^t + 6te^t - (6e^t) = (lnx)^3 x - 3(lnx)^2 x + 6lnx * x - 6x] \ {}_{e^2}^{e^3}$$

$$27x - 3(3^2)x + 6(3)e^3 - 6e^3 - (8(e^2) - 12x + 12x - 6e^2) = 12e^3 - 8e^2 + 6e^2 = 12e^3 - 2e^2$$

15. **A** Find $\frac{dy}{dx}$ when x = 8 if $y = \frac{\sqrt{x^2-3x}}{\sqrt{9-x}}$.

$$y = \frac{\sqrt{x^2-3x}}{\sqrt{9-x}} = \frac{\left(\frac{x^2-3x}{9-x}\right)^{\frac{1}{2}}}{} ; y'$$

$$= \frac{1}{2}\left(\frac{x^2-3x}{9-x}\right)^{-\frac{1}{2}}\left[\frac{\left((2x-3)(9-x)-(-1)(x^2-3x)\right)}{(9-x)^2}\right]$$

$$y'(8) = \frac{1}{2}\left(\frac{64-24}{1}\right)^{-\frac{1}{2}}\left[\frac{\left((13)(1)-(-1)(64-24)\right)}{1}\right] = \frac{1}{2} * \frac{53}{\sqrt{40}} = \frac{53}{4\sqrt{10}} = \frac{53\sqrt{10}}{40}$$

16. **D** Find $\frac{d^2y}{dx^2}$ if $y = \cos^3 x$.

$$y' = 3\cos^2 x\,(-sinx) = -3\cos^2 x sinx = -3(1-\sin^2 x)(sinx) = 3\sin^3 x - 3sinx$$

$$y'' = 9\sin^2 x cosx - 3cosx$$

17. **B** Find all points of inflection of the graph of: $y = x^4 - x^3$.

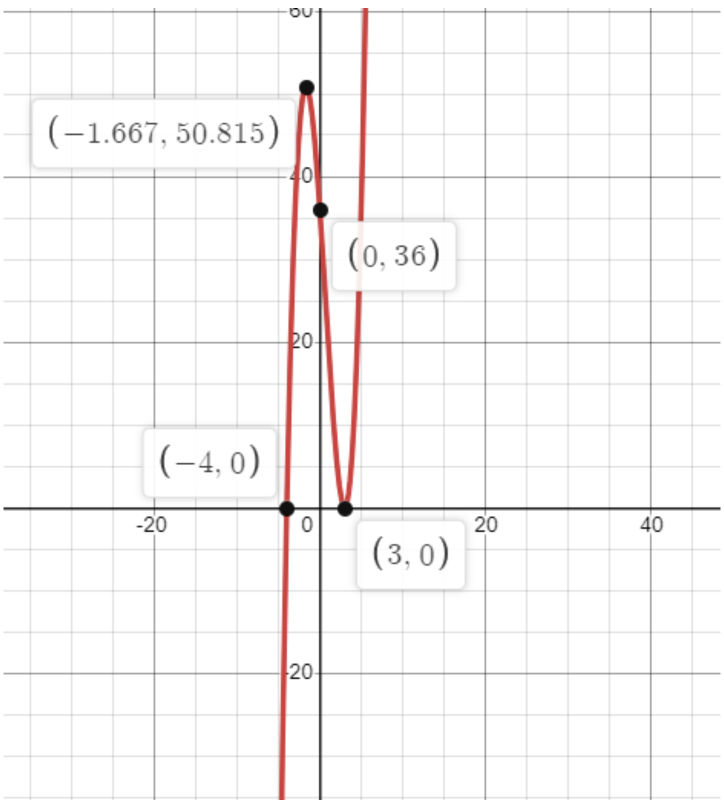
$$y' = 4x^3 - 3x^2; y'' = 12x^2 - 6x;\ 6x(2x-1) = 0;\ x = 0, \frac{1}{2}$$

Plug into the original equation: (0,0) and (1/2, -1/16)

Using the number line you can see that the sign changes from positive to negative to positive again at the two possible points of inflection, which makes them both P.O.I.

(0,0) and (1/2, -1/16)

18. **A** Which statement is NOT true about the graph of: $y = (x-3)^2(x+4)$?



A. There is a relative maximum at (-5/3, 48)
B. The graph is decreasing at the y-intercept of (0,36).
C. There are two x-intercepts at (-4,0) and (3,0).
D. There is a local minimum at the x-intercept (3,0).

19. **B** Find the length of the curve $y = x^{\frac{3}{2}}$ from x = 1 to x = 4.
Arc Length formula:

$$\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$y = x^{3/2}$; $y' = 3/2\, x^{1/2}$

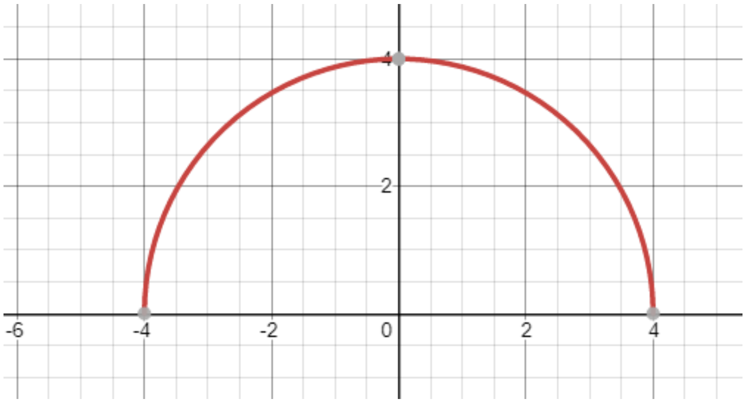
$$L = \int_1^4 \sqrt{1 + [3/2\, x^{1/2}]^2}\, dx =$$

$\int_1^4 \sqrt{1 + 9/4x}\, dx$; $u = 1 + 9/4\, x$; $du = 9/4dx$; $4/9 \int u^{1/2}\, du = 4/9(2/3)u^{3/2}$ $= 8/27(1 + 9/4x)^{3/2}$

$8/27\, [(10)^{3/2} - (13/4)^{3/2}]$ $= 8/27\, [\sqrt{1000} - \sqrt{2197}/8] =$ $80/27\sqrt{10} - \sqrt{2197}/27$

20. **B** Evaluate: $\pi \int_{-4}^{0} \sqrt{16 - x^2}dx$

The graph of this function is a semicircle with radius = 4 centered at the origin, so the area is 1/4(4)2 =4π



21. $\lim_{x\to\infty} \left(\frac{x}{x-1}\right)^{x}$ **B**

$\lim_{x\to\infty} (x/(x-1))^{x} \; u = e^{lnu}$

$\lim_{x\to\infty} e^{ln(x/(x-1))^{x}} = \lim_{x\to\infty} e^{xln(x/(x-1))} = e^{\lim_{x\to\infty} xln(x/(x-1))} = e^{\lim_{x\to\infty} ln(x/(x-1))/(1/x)}$ Indeterminate form, use L'Hopital's

$e^{\lim_{x\to\infty} (-x(x-1))-x/(x-1)^2 \, 1/(x-1))} = e^{\lim_{x\to\infty} x/(x-1)} = e^{\lim_{x\to\infty} 1/(1-1/x)} = e^{1/\lim_{x\to\infty}(1-1/x)} = e^{(1-0)^{-1}} = e$

22. **C** Find the particular solution of the differential equation $\frac{dy}{dx} = -2xy$ given the initial condition $y(3) = 12e^4$.

$dy/y = -2xdx \; ; \; \int dy/y = \int -2xdx \; ; \; ln(y) = -x^2 + c \; ; \; e^{(lny)} = e^{(-x^2+c)}; \; y = Ce^{-x^2}$

$y(3) = e^{-9+c} = 12e^4 \; ; \; Ce^{-9} = 12e^4 \; ; \; C = 12e^{13} \; ; \; y = 12e^{13-x^2}$

23. **C** Given $f(x) = 8\arcsin\left(\frac{x}{20}\right)$, what is $f'(12)$?

Sin y / 8 = x/20

⅛ cos y dy/dx = 1/20

Using Pythagorean Theorem you can solve the triangle to be a 12-16-20 triangle.

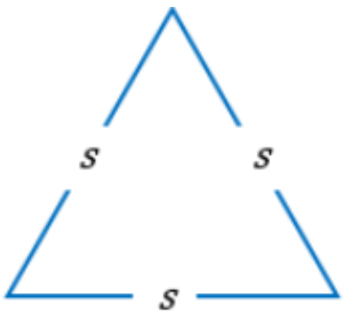
dy/dx = ⅖ sec y

(⅖)(20/16)=½

24. **B** Evaluate using the trapezoidal rule with n = 3: $\int_0^3 x^3 dx$.

T3=½ [f(0)+2f(1)+2f(2)+f(3)]=½ [0+2(1)+2(8)+27]=½ [2+16+27]=45/2 = 22.5

25. **D** A solid has equilateral triangular cross sections perpendicular to the x-axis with a side in the region bounded by $y = x^2$, x = 2 and x = 4. What is its volume?



$$A = \frac{\sqrt{3}}{4} s^2$$

$A = \sqrt{3}/4\ (x^2)^2$; $V = \sqrt{3}/4 \int_2^4 x^4 dx = \sqrt{3}/4\ (1/5)x^5$; $\sqrt{3}/20\ [1024 - 32] = \frac{248\sqrt{3}}{5}$

26. **B** Let $f(x) = |x - 3|$, evaluate: $\lim_{x \to 3^-} f'(x)$.

This is the absolute value graph, where on the interval $(-\infty, 3)$ the graph is decreasing at a constant value of -1 and on the interval $(3, \infty)$ the graph is increasing at a constant rate of 1. -1

27. **A** Evaluate $\lim_{x \to -4^+} \frac{-4x^2+64}{x^2+18x+56}$

$\frac{-4(x^2-16)}{(x+14)(x+4)} = \frac{-4(x+4)(x-4)}{(x+14)(x+4)}$; sub in x = -4 , (-4)(-8)/10 = 3.2

28. **D** Find $\frac{dy}{dx}$ given $y = 2\sqrt{x - \sqrt{x - \sqrt{x - \cdots}}}$.

$y^2 = 4(x - y/2)$; $y^2 + 2y = 4x$; $2y\ dy/dx + 2\ dy/dx = 4$; $dy/dx\ [2y + 2] = 4$; $dy/dx = 4/[2y + 2]$

29. **E** Find the value of the derivative of the graph of the equation at the point when x = 1 of: $36x^2 + y^2 = 144$.

$$36(1)^2 + y^2 = 144; yields\ (1, \pm\sqrt{108})$$

$\frac{36x^2+y^2=144}{144}; \frac{x^2}{4} + \frac{y^2}{144} = 1; \frac{1}{4}(2x) + \left(\frac{1}{144}\right)(2y)\left(\frac{dy}{dx}\right) = 0; \frac{dy}{dx} = -\frac{\frac{1}{2}x}{\frac{1}{72}y}; \frac{dy}{dx}\left(1, \sqrt{108}\right) = -\frac{\frac{1}{2}}{\frac{\sqrt{108}}{72}} = -\frac{36}{\sqrt{108}} =$
$-\frac{6}{\sqrt{3}} = -\frac{6\sqrt{3}}{3} = -2\sqrt{3})$

OR

$\frac{dy}{dx}\left(1, -\sqrt{108}\right) = \frac{\frac{1}{2}}{\frac{\sqrt{108}}{72}} = \frac{36}{\sqrt{108}} = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3})$

30. **C** A continuous function has the data points listed in the table below. Use the information to find the difference of the Left-Hand Riemann Sum and Right-Hand Riemann Sum approximations with n = 6 equal width rectangles on the interval [-14, -2].

x	-2	-4	-6	-8	-10	-12	-14
y	0	3	1	4	2	5	7

R Sum = 2 [0+3+1+4+2+5]=(2)(15)=30
L Sum = 2 [3+1+4+2+5+7]=22(2)=44
Difference = 44-30 = 14
*Note that on a number line, -14 is farther left than -2, the points were given in reverse order.